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ABSTRACT

THIS SERIES OF BOOKLETS IS DESIGNED FOR USE BY STUDENTS OF NINTH GRADE GENERAL MATHEMATICS. THE UNITS ARE THE RESULT OF EXPERIMENTAL WORK DONE BY THE GENERAL MATHEMATICS WRITING PROJECT OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS. THE SERIES TREATS A VARIETY OF TOPICS WHICH ARE SUITABLE FOR USE WITH GENERAL MATHEMATICS STUDENTS. EACH UNIT IS SELF-CONTAINED. THE TITLES OF UNITS 1-5 ARE AS FOLLOWS: UNIT 1: FORMULAS, GRAPHS, AND PATTERNS; UNIT 2: PROPERTIES OF OPERATIONS WITH NUMBERS; UNIT 3: MATHEMATICAL SENTENCES; UNIT 4: GEOMETRY; UNIT 5: ARRANGEMENTS AND SELECTIONS. ALSO INCLUDED IS A BOOKLET CONTAINING CORRECT ANSWERS TO EXERCISES IN UNITS 1-5. (FL)

1

Formulas, Graphs and Patterns

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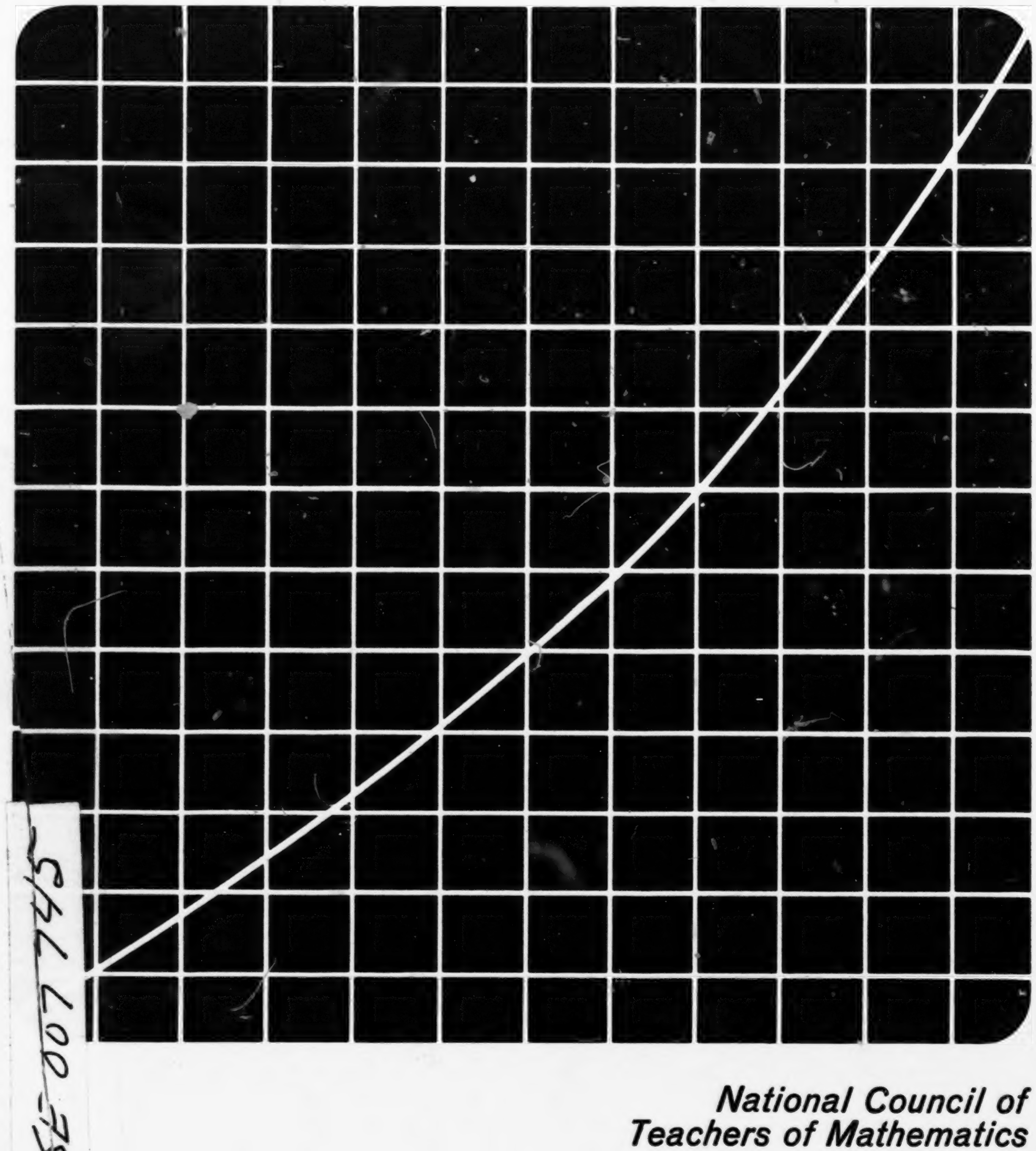

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UNIT ONE OF

Experiences in Mathematical Discovery

Formulas, Graphs, and Patterns

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Preface

"Experiences in Mathematical Discovery" is a series of ten self-contained units, each of which is designed for use by students of ninth-grade general mathematics. These booklets are the culmination of work undertaken as part of the General Mathematics Writing Project of the National Council of Teachers of Mathematics (NCTM).

The titles in the series are as follows:

Unit 1: *Formulas, Graphs, and Patterns*
Unit 2: *Properties of Operations with Numbers*
Unit 3: *Mathematical Sentences*
Unit 4: *Geometry*
Unit 5: *Arrangements and Selections*
Unit 6: *Mathematical Thinking*
Unit 7: *Rational Numbers*
Unit 8: *Ratios, Proportions, and Percent*
Unit 9: *Measurement*
Unit 10: *Positive and Negative Numbers*

This project is experimental. Teachers may use as many units as suit their purposes. Authors are encouraged to develop similar approaches to the topics treated here, and to other topics, since the aim of the NCTM in making these units available is to stimu-

late the development of special materials that can be effectively used with students of general mathematics.

Preliminary versions of the units were produced by a writing team that met at the University of Oregon during the summer of 1963. The units were subsequently tried out in ninth-grade general mathematics classes throughout the United States.

Oscar F. Schaaf, of the University of Oregon, was director of the 1963 summer writing team that produced the preliminary materials. The work of planning the content of the various units was undertaken by Thomas J. Hill, Oklahoma City Public Schools, Oklahoma City, Oklahoma; Paul S. Jorgensen, Carleton College, Northfield, Minnesota; Kenneth P. Kidd, University of Florida, Gainesville, Florida; and Max Peters, George W. Wingate High School, Brooklyn, New York.

The Advisory Committee for the General Mathematics Writing Project was composed of Emil J. Berger (*chairman*), St. Paul Public Schools, St. Paul, Minnesota; Irving Adler, North Bennington, Vermont; Stanley J. Bezuszka, S.J., Boston College, Chestnut Hill, Massachusetts; Eugene P. Smith, Wayne State University, Detroit, Michigan; and Max A. Sobel, Montclair State College, Upper Montclair, New Jersey.

The following people participated in the writing of either the preliminary or the revised versions of the various units:

Donald H. Firl, Rochester Public Schools, Rochester, Minnesota

Carroll E. Fogal, University of Florida, Gainesville, Florida

Edward F. Gottlieb, Madison High School, Portland, Oregon

Kenneth P. Kidd, University of Florida, Gainesville, Florida

Edna K. Lassar, Polytechnic Institute of Brooklyn, Brooklyn, New York

John F. LeBlanc, Racine Public Schools, Racine, Wisconsin

Scott D. McFadden, Spencer Butte Junior High School, Eugene, Oregon

Max Peters, George W. Wingate High School, Brooklyn, New York

Leonard Simon, Bureau of Curriculum Research, New York Public Schools, New York, New York

Irwin N. Sokol, Collinwood High School, Cleveland, Ohio

William H. Spragens, University of Louisville, Louisville, Kentucky

Mary E. Stine, Fairfax County Schools, Fairfax, Virginia

Arthur J. Wiebe, Pacific College, Fresno, California

John E. Yarnelle, Hanover College, Hanover, Indiana

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Finally, a word of grateful thanks is extended to Julia A. Lacy and Charles R. Hucka for preparing the copy for the printer, and to James D. Gates for attending to financial matters.

EMIL J. BERGER
Chairman, Advisory Committee
General Mathematics Writing Project

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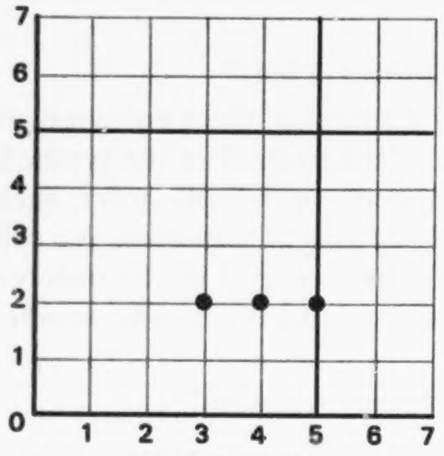
Experiences in Mathematical Discovery

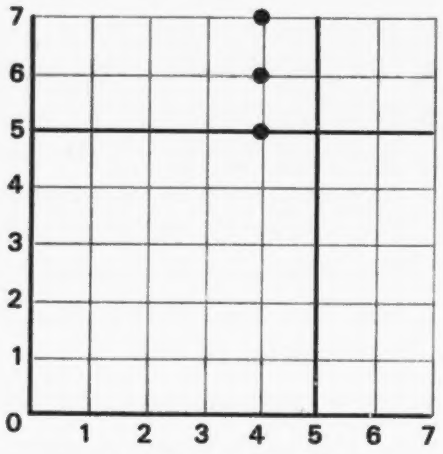
Formulas, Graphs, and Patterns

1 Battleship

Perhaps some of you have played the game of battleship. John and David have decided to make up their own version of the game. They agree to represent a battleship by three dots placed next to each other either in a row or in a column at the intersections of the grid lines on a piece of squared cross-section paper. The charts below show how each boy has hidden his battleship. Neither boy knows where the other has located his battleship, and in playing the game neither boy is permitted to see the other's chart. The object of the game is to sink the other person's battleship. Let us see how the game is played.



John's Battleship



David's Battleship

Class Discussion 1

1. It is decided that John, the older of the two boys, will take the first turn. John is allowed three "shots" at David's battleship.

First shot: (5 to the right, 3 up)
Second shot: (7 to the right, 1 up)
Third shot: (4 to the right, 4 up)

David looks at his chart and tells John that he missed on all three shots. Do you agree?

2. Now it is David's turn to shoot at John's battleship. (It is agreed that to sink a battleship one must make three successful shots.)

First shot: (2 to the right, 1 up)
Second shot: (4 to the right, 2 up)
Third shot: (1 to the right, 1 up)

John admits that David has made one hit and that his battleship is damaged. The boys agree that it is not necessary to tell which shot has been successful. Do you agree that there has been one successful shot? If so, which one?

3. It is John's turn again.

First shot: (4, 5)
Second shot: (5, 4)
Third shot: (2, 2)

The boys agree to use the symbol (4, 5) as an abbreviation for (4 to the right, 5 up).

a. Does (4, 5) indicate the same position as (5, 4)?

b. Does the *order* of the numbers named in the two symbols make a difference in the position of the point that the symbols represent? A pair of numbers arranged in a definite order, such as (4, 5) or (5, 4), is called an *ordered pair* of numbers. We call the number that is named first the *first number* of the ordered pair and the number that is named second the *second number* of the ordered pair.

c. Does John have a hit? If so, which shot was a hit?

4. For his next turn David calls the following ordered pairs of numbers: (1, 7), (7, 1), (7, 7). John is surprised. He remarks that David has just wasted three shots.

  a. Do you agree with John? Why?

  b. How can the boys keep a record of the shots they have taken?

5. John next calls the following ordered pairs of numbers: (3, 5), (5, 5), (5, 6).

  a. Do these ordered pairs of numbers represent sensible shots? Why?

  b. Does John have any hits?

6. David is careful not to waste any more shots. In fact, on his next turn he sinks John's battleship. What ordered pairs of numbers did he need?

7. The numbers in the ordered pairs, which give the location of a point, are called *coordinates* of the point. What are the coordinates of the points that represent David's battleship?

8. The point associated with the ordered pair of numbers (0, 0) has a special name. It is called the *origin*. In the game just completed, did either John or David use the origin in hiding his ship?

9. The two lines in the chart that intersect at the origin are called *coordinate axes*. The line that is parallel to the top and bottom edges of this book is referred to as the *horizontal axis*, and the line that is parallel to the side edges is referred to as the *vertical axis*. Which of the ordered pairs of numbers named below are associated with points that are on the horizontal axis? On the vertical axis? Which ordered pairs are associated with points that are on neither of the coordinate axes?

(5, 0), (0, 6), (3, 3), (0, 1), (4, 1), (0, 0), (2, 0)

10. In the battleship game described above, the numbers in the ordered pairs that were used to locate ships and call shots were whole numbers. Recalling that the boys confined their game to a chart that was 7 units square, tell how many ordered pairs of whole numbers they had to choose from in calling their shots.

11. In the next game that the boys play they do not restrict themselves to ordered pairs of whole numbers in hiding their battleships, and John hides his ship on points associated with the following ordered pairs of numbers: (3½, 2¾), (4½, 2¾), (5½, 2¾). Do these points lie in a horizontal line or in a vertical line? Can you answer the last question without actually locating the points that are associated with these ordered pairs?

12. After several turns, David becomes discouraged. What do you think are David's chances of sinking John's battleship?

13. If fractions are used in a game of battleship, how many possible ordered pairs are there if the game is played on a chart that is 7 units square?

Summary—1

1. Agreements or rules are needed in playing a game such as battleship. The battleship games that have been described show how ordered pairs of numbers may be associated with points on a chart. The chart here used is simply a piece of squared cross-section paper. The network of lines on the chart is called a grid.

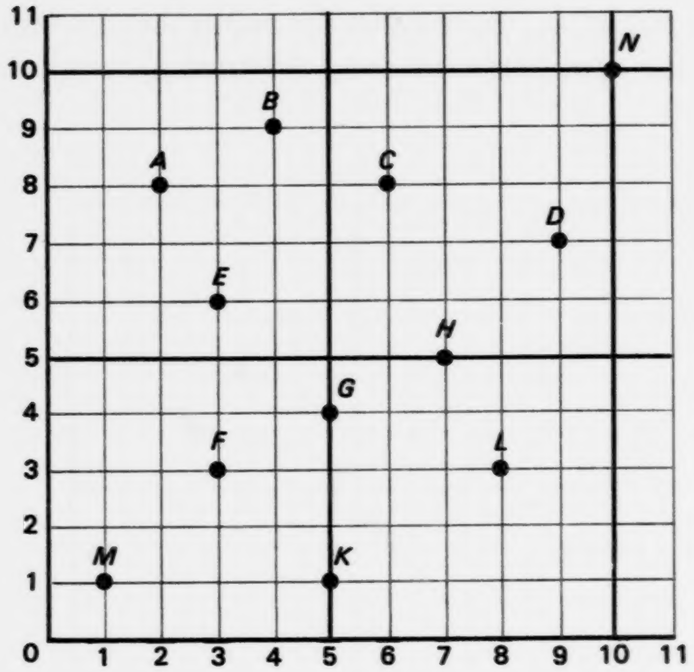
2. What does it mean to say that an ordered pair of numbers is associated with a point on a chart? It means that the ordered pair of numbers has been *assigned* to the point or that the point has been *assigned* to the ordered pair of numbers. The assignment works both ways.

3. To assign ordered pairs of numbers to points on a chart, or to assign points on a chart to ordered pairs of numbers, we need to have two reference lines on which scales that measure distances are drawn. The two reference lines are usually called *coordinate axes*. It is customary to choose a line extending from left to right for one of these axes, namely, the *horizontal axis*, and to choose a line extending up and down for the second or *vertical axis*. The point of intersection of the two axes is called the *origin*. The ordered pair of numbers assigned to the origin is (0, 0). The scale to be drawn on each coordinate axis is needed to measure distances from the origin.

4. When we plot the point associated with an ordered pair of numbers, the first number tells us how many units from the vertical axis the point is located. The second number tells us how many units from the horizontal axis the point is located. To mark the position of a point, we usually make a dot. The dot is the *graph* of the ordered pair of numbers associated with the point.

Exercises—1

1. Twelve points are plotted on the chart below. What are the coordinates of each of these points?



2. Plot the points associated with the ordered pairs given below. If the points are plotted correctly, the collection of dots that you get will suggest a pattern that is familiar to you.

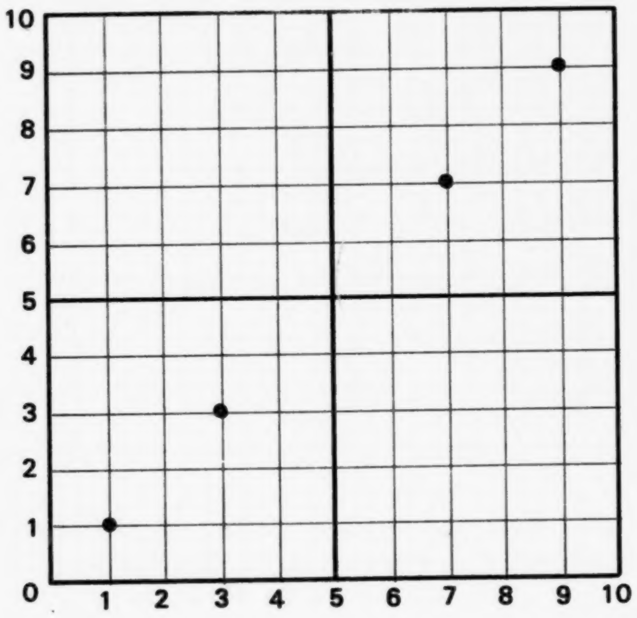
(5, 5), (7, 3), (7, 11), (3, 5), (7, 12½)
(12, 5), (11, 3), (3½, 4), (7, 9½), (10, 5)
(13½, 5), (7, 5), (12, 3½), (8, 5), (7, 6½)
(8, 3), (5, 3), (7, 8), (9, 3), (13, 4)

In this unit, and in others, we shall often use the word *pattern*. In mathematics there are many situations in which one can observe patterns. For example, the set of numbers in the following sequence suggests a pattern. What is the pattern?

2, 5, 8, 11, 14, 17

The pattern in this case is a number pattern. Certain sets of points may also suggest patterns depending on the way in which the points are arranged. Such patterns would be geometric patterns.

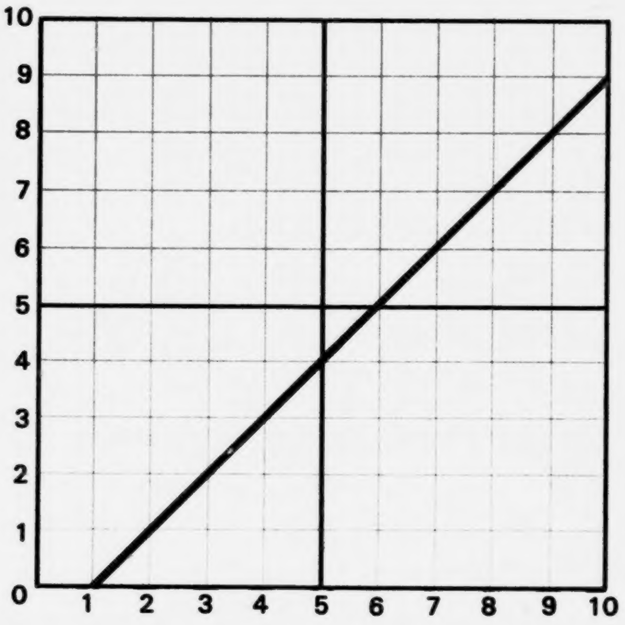
3. **a.** List the ordered pairs of numbers that are associated with the points indicated by dots on the grid below.
b. What pattern is suggested by the points that are indicated?
c. Give the coordinates of some additional points that fit this pattern.



4. List five ordered pairs of numbers in each of which the first number is two less than the second number. Plot the points that are associated with these ordered pairs.

5. List five ordered pairs of numbers in each of which the second number is four times the first number. Plot the points that are associated with these ordered pairs.

6. a. The graph shown below is a straight line. Choose five points that are on this graph, and give their coordinates.
   b. If the chart were larger, the line could be extended upward to the right. List the coordinates of five points that would be on the extended line.



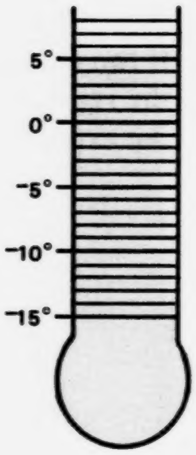
7. All points that have been plotted so far have been plotted in two dimensions—that is, on a flat sheet of paper. John and David decide to plot points in *three* dimensions. Now, *ordered triples* instead of ordered pairs are used to locate points. Also, instead of using only two coordinate axes the boys need to use three coordinate axes.
   a. Can you suggest a way of setting up three coordinate axes?

b. Suppose that the point associated with (3, 4, 7) is to be plotted. How do you suppose this point can be located?

c. How can the point associated with (5, 2, 11) be located?

2 Plotting a Cold Night

Jeff lives in northern Wisconsin. One day in January on his return from school he heard the 4:00 P.M. weather forecast. The radio announcer said that the temperature at that time was 6 degrees. His prediction was that for the next seven hours the temperature would fall at the rate of 2 degrees each hour. Jeff was working on a weather project in his science class. He decided to include this information in his report.

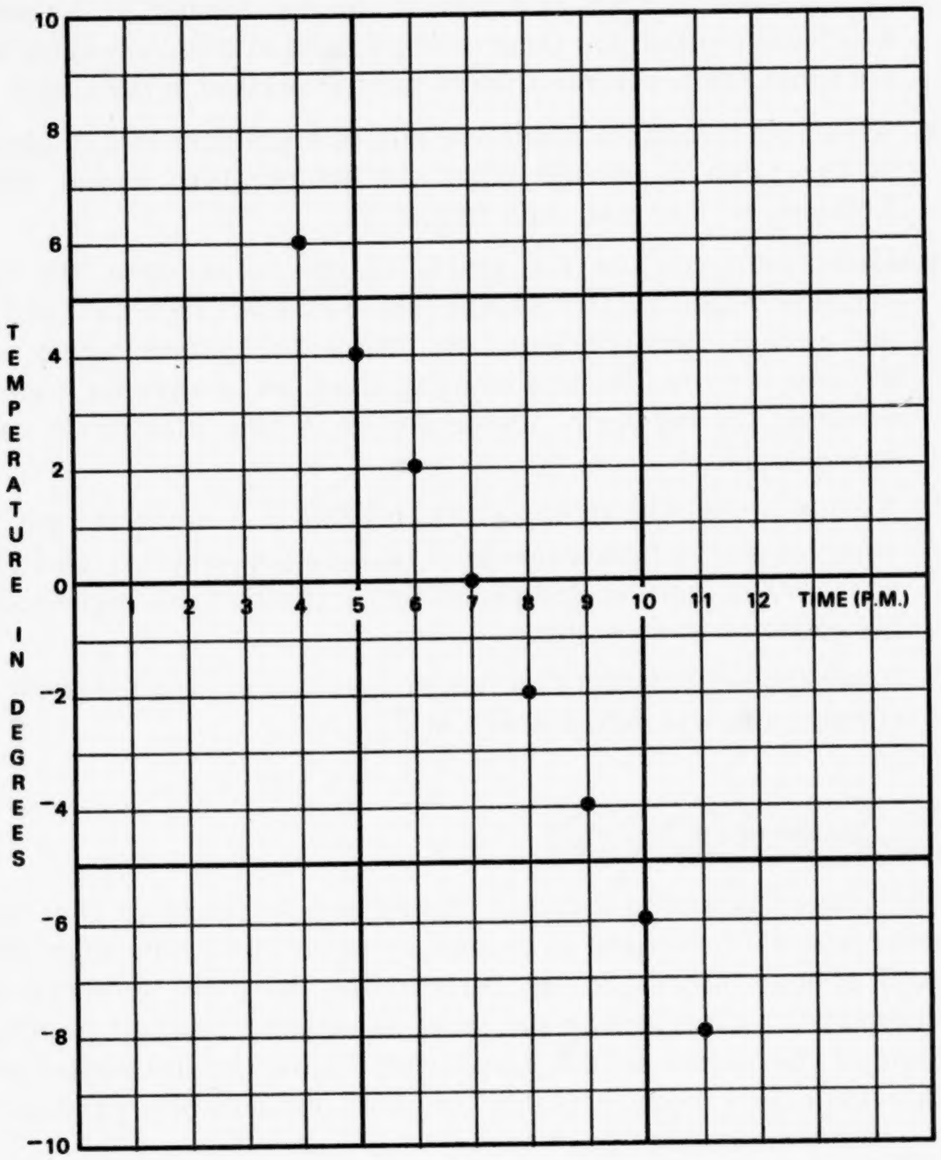


Class Discussion 2

1. According to the forecast, at what time would the temperature be zero degrees?
2. According to the forecast, what would the temperature be at 9:00 P.M.?
3. The temperature prediction for 10:00 P.M. was 6 degrees below zero. Another way of writing "6 degrees *below* zero" is $^-6^\circ$. This symbol is read "negative six degrees." According to the forecast, what would the temperature be at 11:00 P.M.?
4. Jeff decided to make a *table of values* for the time and temperature. Copy and complete the table.

Time	4:00 P.M.	5:00 P.M.	6:00 P.M.	7:00 P.M.	8:00 P.M.	9:00 P.M.	10:00 P.M.	11:00 P.M.
Temperature	6°							

Later Jeff made a graph of the temperature predictions. In making the graph he interpreted the ordered pairs (time, temperature) expressed in the table as ordered pairs of numbers. For example, he interpreted (4:00 P.M., 6°) as (4, 6).



5. According to the graph, what would the temperature be at 6:00 P.M.? Does this reading agree with the temperature recorded in the table?

6. The point associated with the ordered pair (5, 4) is five units to the right of the vertical axis and four units above the horizontal axis. Describe the position of the point associated with the ordered pair (8, ⁻2).
7. List, as ordered pairs of numbers, the coordinates of the points which are plotted. Do these ordered pairs of numbers agree with the (time, temperature) ordered pairs expressed in the table?
8. Assuming that the temperature falls at a uniform rate, make use of the table to decide what the temperature would be at 5:30 P.M. At 5:15 P.M. At 5:45 P.M.
9. How could you use the graph to determine what the temperature would be at 5:30 P.M.? At 5:15 P.M.? At 5:45 P.M.? Do the answers you obtained by using the graph agree with the answers you obtained by using the table in exercise 8 above? Finding "in between" temperatures in this manner is called *interpolation*.
10. Assuming that the temperature did fall at a uniform rate, can you plot all the points for the time between 4:00 P.M. and 11:00 P.M.? What familiar geometric figure (pattern) is suggested by the graph of these points?
11. From the information that has been given can you tell what the temperature was before 4:00 P.M.?

Summary—2

In Class Discussion 2 it was shown how to organize time-and-temperature information in a *table of values*, and how to make a graph of such information by interpreting the (time, temperature) ordered pairs expressed in the table as ordered pairs of numbers. Some of the points on the graph were below the horizontal axis. This happened because some of the temperature predictions involved temperatures below zero.

We noted a *pattern* in the graph of the temperature predictions. All points appeared to be in a straight line. By assuming that this was indeed the case we *interpolated* between the points that had been plotted and made various predictions of our own. For example, we used the graph to determine what the temperature might be at 5:45 P.M. More attention will be given to the reading of graphs in the lessons that follow.

Exercises—2

1. The next day Jeff called the weather bureau to get the actual temperature readings of the previous evening. These are entered in the table below. Make a graph showing the information in the table.

Time	Temperature
4:00 P.M.	6°
5:00 P.M.	5°
6:00 P.M.	5°
7:00 P.M.	2°
8:00 P.M.	$^-1$°
9:00 P.M.	$^-5$°
10:00 P.M.	$^-9$°
11:00 P.M.	$^-12$°

2. One cold morning the temperature reading at 6:00 A.M. was 17 degrees below zero. The forecaster predicted that the temperature would rise to 8 degrees above zero by 12:00 noon.

a. Make a graph of the two points that are indicated by the given information.

b. Suppose that the temperature continued to rise at the same rate from 6:00 A.M. until 12:00 noon. Use your graph to help you determine the approximate temperatures at 7:00 A.M., 8:00 A.M., 9:00 A.M., 10:00 A.M., and 11:00 A.M.

c. Again, assume that the temperature continued to rise at the same rate. At approximately what time would the temperature be $^-13$°? $^-5$°? 0°? 6°?

3. Consider the points plotted on each chart shown below.

a. What ordered pairs of numbers are associated with the points plotted in A?

b. What ordered pairs of numbers are associated with the points plotted in B?

c. Study your answers to exercises 3a and 3b. What conclusion, if any, can you state?

d. The two graphs shown below do not look alike, but each represents the same information. What reason can you give for the fact that the graphs are different in appearance?

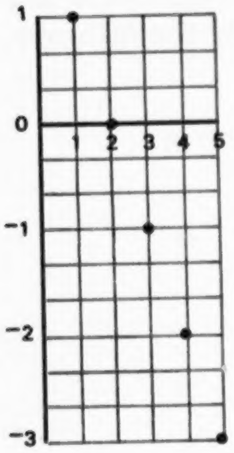


FIG. A

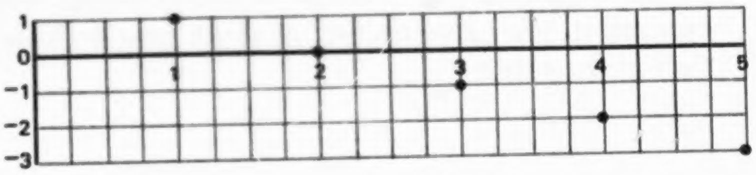


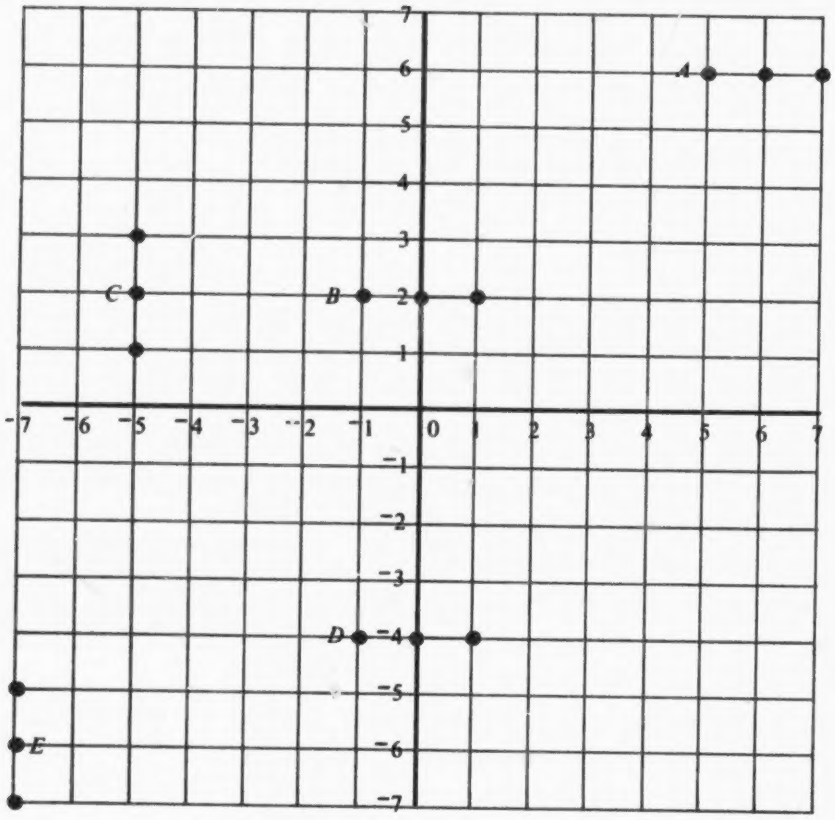
FIG. B

4. Plot on a chart the points associated with the ordered pairs of numbers listed below. If the points are plotted correctly, they will be on a straight line.

(1, ⁻24), (10, 3), (6, ⁻9), (12, 9)
(3, ⁻18), (7, ⁻6), (16, 21)

5. At what point does the line suggested in exercise 4 intersect the horizontal axis? The vertical axis?

6. Consider again the battleship game described in the previous section. Suppose that the two boys had decided to extend their game of battleship, and had agreed that each would hide five battleships instead of one. Suppose further that they had decided to extend the grid used in the game so that it would look like the one shown below. What are the coordinates of the points that indicate the positions of the five ships?



7. a. Plot the following points on a chart. (Note that we are referring to ordered pairs of numbers as points. This is acceptable because points are associated with ordered pairs of numbers.)

(⁻3, ⁻4), (5, 0), (⁻4, 3), (3, ⁻4), (4, ⁻3), (0, 5)
(0, ⁻5), (⁻5, 0), (4, 3), (2, 4.6), (⁻4.6, ⁻2)

b. What geometric figure is suggested by the points that you plotted in exercise 7a?

8. a. If the first number in an ordered pair is negative, is the point associated with the ordered pair located to the left or to the right of the vertical axis?

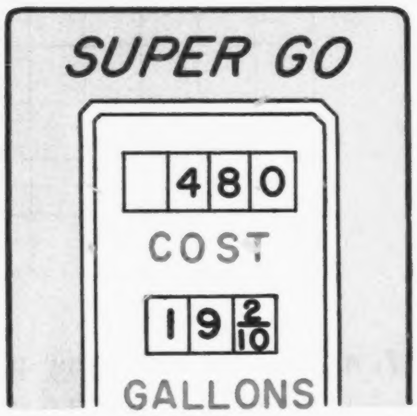
b. If the second number in an ordered pair is negative, is the point associated with the ordered pair located above or below the horizontal axis?

9. When the following points are plotted, a familiar letter of the alphabet is suggested by the graph. List ordered pairs for five additional points that will fit this pattern.

$$(^{-}3, 7), (^{-}3, 5), (^{-}3, 2), (^{-}3, ^{-}1)$$
$$(^{-}3, ^{-}4), (^{-}1, ^{-}4), (2, ^{-}4)$$

3 At the Service Station

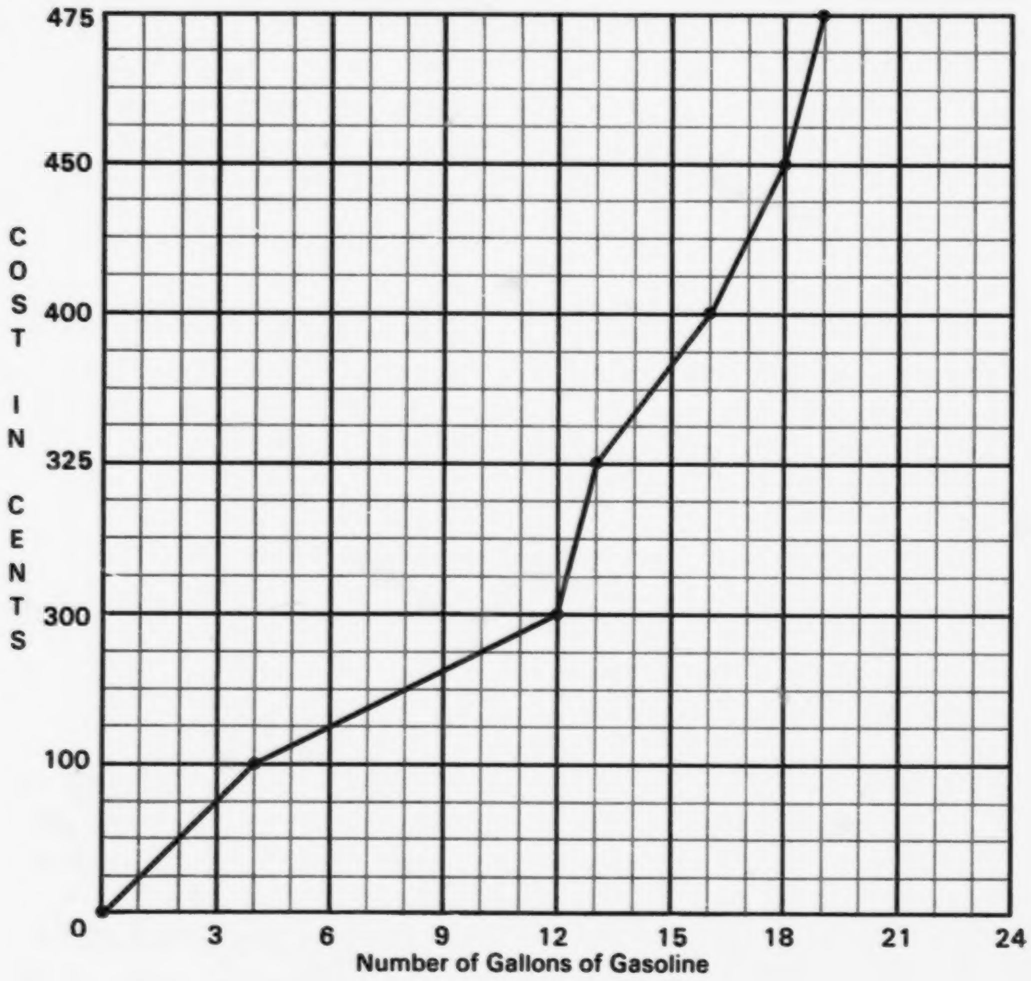
Mr. Olson is a mathematics teacher at Central Junior High School. On his way to school one morning he noticed that his gas tank was nearly empty, so he drove to the nearest service station.



The picture at the right shows the number of gallons added to his tank and the amount that he was charged. How much gasoline was added to his tank? How much was he charged?

Mr. Olson observed that the service station attendant kept a record of each sale by recording the number of gallons of gasoline sold and the number of cents that the gasoline cost the purchaser. A record such as the attendant kept is called a table of values. (See exercise 4, Class Discussion 2.) Note that the entries in each row of the table can be thought of as ordered pairs of numbers of the form (number of gallons, cost in cents). Mr. Olson decided to share with his class the information in the table. When he arrived at school, he drew on the chalkboard the graph shown below. Some students reacted unfavorably to this graph. They felt it was *not* a good picture of the information. Together, Mr. Olson and the class discussed ways of making improvements.

Number of Gallons	Cost in Cents
12	300
4	100
18	450
13	325
16	400
19	475



Class Discussion 3

To answer the following questions, you will need to make use of the graph that Mr. Olson drew on the chalkboard.

1. How many gallons of gasoline does the length of each unit segment on the horizontal axis represent?
2. Do all unit segments on the vertical axis represent the same number of cents? Explain.
3. Make use of the graph to determine the cost of 5 gallons of gasoline.

4. Is it convenient to use the graph to determine the cost of 15 gallons of gasoline?
5. If one were to look at the graph quickly, why might he conclude that 12 gallons of gasoline costs twice as much as 4 gallons?
6. How would you explain the fact that the appearance of the graph suggests that 19 gallons of gasoline costs twice as much as 13 gallons?
7. Notice that the upper part of the graph is steeper than the lower part. Does this mean that the price per gallon increases as the number of gallons purchased increases? Why do you think the upper part of the graph is steeper than the lower part?
8. What do you consider the main weakness of this graph?
9. What suggestions can you make for improving this graph?

Summary—3

A graph should be easy to read. Yet we found some difficulty in reading the graph that Mr. Olson drew on the chalkboard. The difficulty was due to irregularities in the scale chosen for the vertical axis.

It is not necessary for the vertical axis and the horizontal axis to have the same scale. However, it is important that each scale be uniform. For example, if one unit segment on the horizontal axis is to represent one gallon, then each unit segment on this axis must represent one gallon. Likewise, if one unit segment on the vertical axis is to represent 25 cents, then each unit segment on this axis must represent 25 cents.

Exercises—3

1. Consider again the information which the service station attendant gave Mr. Olson. Make a graph to represent the infor-

mation. Be sure that each axis has a uniform scale. Without lifting your pencil, draw the line suggested by the points.

2. Use the graph that you made in exercise 1 to complete the following table.

Number of Gallons	5	15		11				1
Cost in Cents			175		425	225	75	

3. Use the graph that you made to find an *approximation* of the cost for each of the following quantities of gasoline.

a. 10.5 gallons
b. 13.2 gallons
c. 19.5 gallons

4. Approximately how much gasoline can be purchased with each of the following amounts of money?

a. 90 cents
b. 235 cents
c. 510 cents

5. Find two points on the graph whose coordinates are *not* included in the original table of values and give the coordinates of these points.

6. If possible, find an ordered pair of numbers in the original table which is not associated with a point on your graph.

7. Which of the following rules can be used to determine the cost of gasoline from the pump used to fill Mr. Olson's tank? In each rule, G represents the number of gallons purchased, and C represents the cost in cents.

$$C = 30 \times G.$$
$$C = 25 + G.$$
$$C = 20 \times G.$$
$$C = 25 \times G.$$

8. How can you be reasonably certain that the rule you selected is the correct one to use?

9. The four ordered pairs of numbers in the table at the right represent four purchases from a different pump, containing Super-Super-Go gasoline. Write a rule that gives C, the cost in cents, of G gallons.

Number of Gallons	Cost in Cents
6	186
11	341
8	248
17	527

4 Graphs and Formulas

Rules like those used above to determine the cost of different quantities of gasoline are called *formulas*. Consider the following formulas:

$$C = 30 \times G.$$

$$C = 30 \cdot G.$$

$$C = 30G.$$

Do we have three different formulas or just one? The answer is that we have just one formula because 30 and G have just one product even though we may use different symbols to indicate it.

A formula is really a *mathematical sentence*. It is not possible to tell whether a mathematical sentence such as $C = 30G$ is true or false until C and G are replaced by numbers. The replacements that are of special interest to us are those that make the mathematical sentence true.

One day Mr. Olson asked his students to think of situations in which the formula $C = 30G$ could be used. One student stated that this formula could be used to find the amount C paid for a certain number of gallons of gasoline if the cost per gallon were 30 cents. If such an interpretation were used, what would the number represented by G indicate?

Class Discussion 4

1. Can you think of other situations in which the formula $C = 30G$ could be used? Can you think of one such situation in which G need not be a whole number?

2. Consider again the formula $C = 30G$.
   a. If G is replaced by 0, what replacement for C in the formula results in a true sentence?
   b. If G is replaced by 1, what replacement for C in the formula results in a true sentence?
   c. If G is replaced by 2, what replacement for C in the formula results in a true sentence?
   d. If G is replaced by 4, what replacement for C in the formula results in a true sentence?
   e. If G is replaced by 7, what replacement for C in the formula results in a true sentence?
   f. If G is replaced by 9, what replacement for C in the formula results in a true sentence?

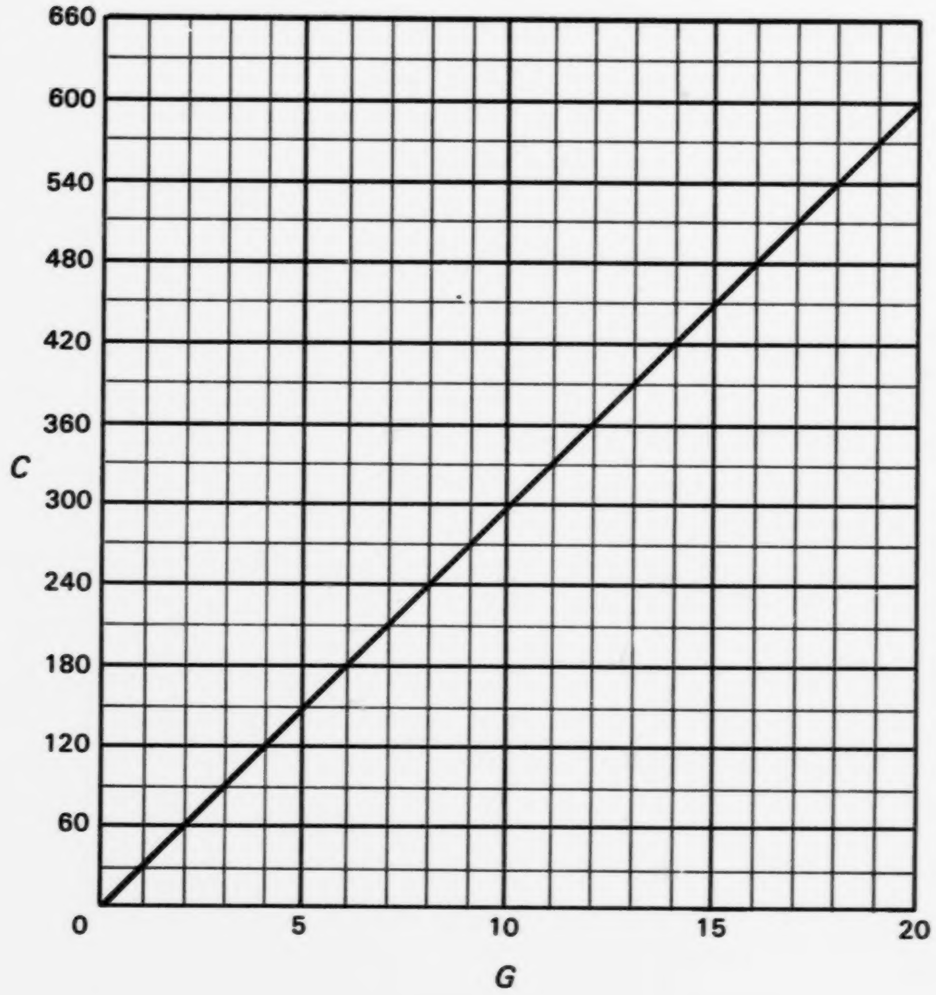
3. A convenient way of recording the information obtained in exercise 2 is to make a table of values for the formula $C = 30G$. Copy and complete the table.

G	0	1	2	4	7	9
C						

4. Each ordered pair of numbers in the completed table is a *solution* of the formula. Another way of stating this is to say that each ordered pair of numbers in the table satisfies the formula. Can you find other ordered pairs of numbers that also satisfy the formula? If so, name some.

5. Give two examples of ordered pairs of numbers that do not satisfy the formula. Are these solutions of the formula?

6. Let us assume that the formula $C = 30G$ represents the cost of gasoline at 30 cents per gallon and that the graph below is a graph of this formula.



a. Does each ordered pair of numbers in your table of values correspond to a point on the graph?

b. Is the point associated with (10, 300) on the graph? Does this ordered pair satisfy the formula? Is this ordered pair in your table of values?

c. Is the point associated with (15, 450) on the graph? Does this ordered pair satisfy the formula? Could this ordered pair have been included in your table of values?

d. Are *all* points on the graph solutions of the formula?

e. Is the point associated with (12, 330) on the graph? Is this ordered pair a solution of the formula? Is the point associated with (5, 180) on the graph? Is this ordered pair a solution of the formula?

7. Use the graph to help you complete the following table.

G	3	6			12	
C			240	330		600

Summary—4

The symbols C and G in the formula $C = 30G$ are called *variables*. Variables may be replaced by numbers. When both C and G in the formula $C = 30G$ are replaced by numbers, the resulting sentence is either true or false. An ordered pair of numbers of the form (replacement for G, replacement for C) which satisfies the formula is called a solution of the formula. To keep a record of the different solutions that we obtained for the formula $C = 30G$, we made a table of values. To make a graph of the formula, we plotted the points associated with the solutions collected in the table of values. (Usually a small number of solutions is sufficient to suggest what the pattern of a graph is.) We found that the graph of the formula $C = 30G$ is a line.

Three important facts concerning the graph of a formula should be noted.

1. For every solution of the formula there is a point on the graph.
2. If an ordered pair of numbers is not a solution of the formula, then the point associated with that ordered pair is not on the graph.
3. If a point is not on the graph, then the ordered pair of numbers associated with the point is not a solution of the formula.

Exercises—4

1. The average speed of John Glenn's space capsule during the historic flight around the earth was approximately 6 miles per second.

a. What is the approximate distance that the capsule traveled in 1 second? In 2 seconds? In 10 seconds?

b. Consider the formula $D = 6t$, where D is a variable for the distance (in miles) that the capsule traveled and t is a variable for the time in seconds. Use this formula to complete the table of values below.

t	0	1	2	3	5	7	10
D	0	6					

c. Make use of the table of values in 1b to draw a graph of the formula $D = 6t$. Choose the horizontal axis as the t-axis. (What this means is that we shall use the values of t to represent distances from the origin measured along the horizontal axis.) Let the vertical axis be the D-axis. Begin the graph by plotting the points corresponding to the ordered pairs of numbers in the table. Then draw the simplest line that connects these points in order from left to right.

d. According to your graph, what is the distance that the capsule traveled in 4 seconds? 6 seconds? 8 seconds? 11 seconds?

e. According to your graph, approximately how many seconds did it take the capsule to travel 9 miles? 21 miles? 54 miles?

f. Use either the formula $D = 6t$ or the graph of the formula to determine how many seconds it took the capsule to travel each of the following distances: 90 miles, 150 miles, 300 miles, 225 miles.

2. Copy and complete the table of values for each of the following formulas.

$C = 3N.$

a.

N	0	1				15			
C			9	12	30		63	90	100

$$A = B + 10.$$

b.

B	0	5			33	41			
A			25	37			100	171	201

$$R = T - 15.$$

c.

T	15	25			80	100		
R			20	35			100	200

$$X = \frac{A}{10}.$$

d.

A	10	20	40	70		
X					8	10

$$X = \frac{1}{10}A.$$

e.

A	10	20	40	70		
X					8	10

3. **a.** Draw a graph of the formula given in exercise 2b. (It is not necessary to plot all the ordered pairs of numbers given in the table.) Let the horizontal axis be the B-axis.

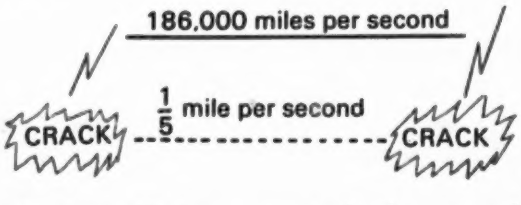
b. Draw a graph of the formula given in exercise 2d. Let the horizontal axis be the A-axis.

c. Describe the graph of the formula given in exercise 2e. Do not draw the graph.

4. Each of the formulas in exercise 2 contains exactly two variables. Many formulas contain more than two variables. For example, the formula $W = \frac{L \times G \times G}{670}$ contains three variables. This formula can be used to find an approximation of the weight in pounds of a freshly caught salmon. In this formula, L represents the length of the salmon in inches and G represents the maximum girth in inches. (The maximum girth is the distance around the thickest part of the fish.) If the length of the salmon is 30 inches and its maximum girth is 22 inches, find the weight of the salmon.

5 Lightning and Thunder

Henry was camping with some friends in the mountains. As sometimes happens, a thunderstorm hit their camp. This forced Henry to remain in his tent. To amuse himself, he decided to collect data about the time that elapses between the moment he sees a flash of lightning and the moment he hears the thunder. From his study of science Henry recalled that light travels 186,000 miles per second and that sound travels $\frac{1}{5}$ mile per second Since the speed of light is extremely fast, Henry decided to assume that it takes no time at all for the light given off by a flash of lightning to reach him. On the other hand, since sound travels much more slowly, he thought that the time it takes for the sound of the thunder which accompanies a flash of lightning to reach him might be measurable.



Henry reasoned that if he counted the number of seconds between the moment that he saw a flash of lightning and the moment that he heard the thunder, he would be able to tell how far away the lightning was. To determine how far away a flash of lightning was, Henry used the formula $D = \frac{1}{5}s$. In this formula D represents Henry's distance from the lightning in miles, and S represents the time in seconds for the sound to reach him.

Class Discussion 5

1. Ten seconds elapsed between the time that Henry saw a flash of lightning and the time that he heard the clap of the thunder. How far away was the lightning?

2. Copy and complete the table below. Find D by using the formula $D = \frac{1}{5}s$.

Time in Seconds (s)	1	3	5	8	10
Distance in Miles (D)					

3. Consider the formula $5D = s$.
   a. If s is 1, what replacement for D in the formula results in a true sentence?
   b. If s is 3, what replacement for D in the formula results in a true sentence?
   c. If s is 5, what replacement for D in the formula results in a true sentence?
   d. If s is 8, what replacement for D in the formula results in a true sentence?
   e. If s is 10, what replacement for D in the formula results in a true sentence?
   f. Record the results obtained above in a table of values and compare the table you get with the table you completed in exercise 2.

4. Two formulas are *equivalent* if the table of values for each formula may be used as a table of values for the other.
   a. Do you think that $5D = s$ and $D = \frac{1}{5}s$ are equivalent?
   b. Which one of the following formulas is not equivalent to $y = 10 + x$?

$y = x + 10.$
$y - 10 = x.$
$y - x = 10.$
$y + 10 = x.$

5. Consider the four graphs shown below in figures A, B, C, and D. Note that the scales have not been indicated on either coordinate axis in any of the figures. Which one of the figures contains the most accurate representation of the graph of $D = \frac{1}{5}s$?

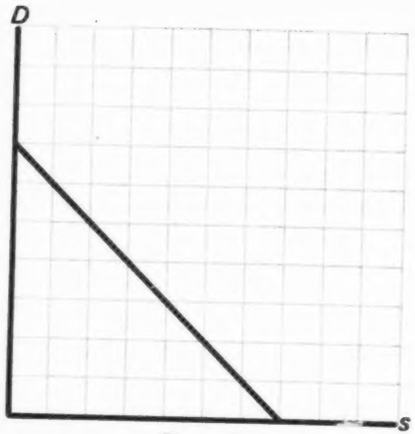


FIG. A

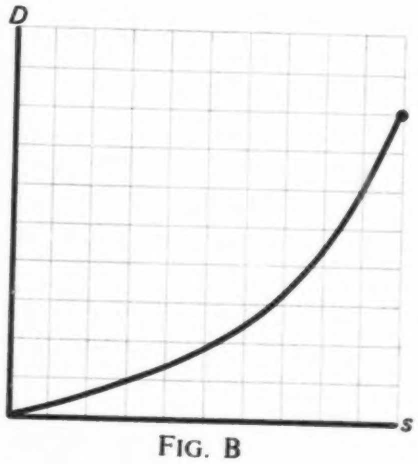


FIG. B

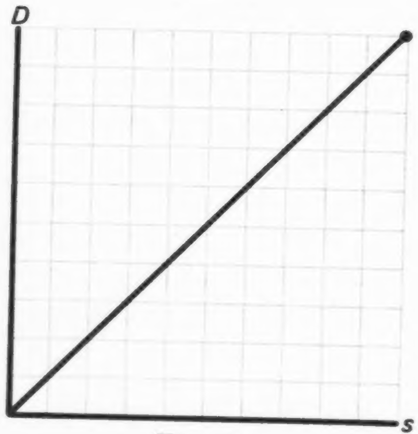


FIG. C

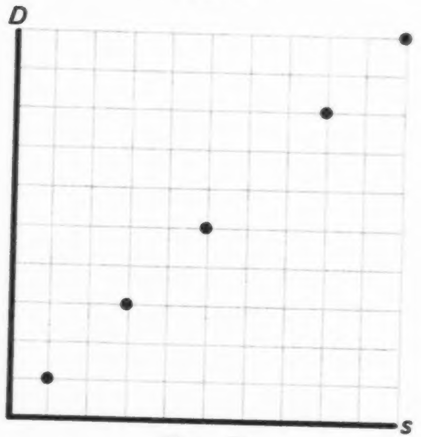


FIG. D

6. Consider again the figure that you feel contains the most accurate representation of the graph of $D = \frac{1}{5}s$. Assign an appropriate scale to the horizontal axis and also to the vertical axis.

7. The graph in figure A of exercise 5 is described as one that "falls to the right." On the other hand, the graphs in figures B, C, and D are said to "rise to the right."

a. Without drawing the graph of $D = \frac{1}{5}s$, how could you tell whether or not the graph rises to the right or falls to the right?

b. What property does a table of values have if its graph rises to the right? If it falls to the right?

Summary—5

The following formulas were discussed in the exercises above: $5D = s$ and $D = \frac{1}{5}s$. Since the table of values of each formula is also the table of values for the other, we say that the formulas are *equivalent*. The two formulas have the same graph. The graph is a line that rises to the right.

Exercises—5

1. Study the tables of values below. Match each table with two of the formulas listed at the right.

A

x	0	1	2	3	4	5
y	3	4	5	6	7	8

B

x	3	4	5	6	7	8
y	0	1	2	3	4	5

C

x	0	1	2	3	4
y	0	3	6	9	12

D

x	3	6	9	12	15
y	1	2	3	4	5

a. $y = x + 3$.
b. $y = \frac{x}{3}$.
c. $y = x - 3$.
d. $y = 3x$.
e. $\frac{y}{3} = x$.
f. $y - 3 = x$.
g. $y + 3 = x$.
h. $3y = x$.

2. Which of the formulas listed in exercise 1 are equivalent?

3. Draw the graph of each formula listed in exercise 1. Choose the horizontal axis as the x-axis. Determine which graphs rise to the right and which ones fall to the right.

4. In this exercise you are asked to draw the graphs of several formulas. Tables of values for some of the formulas are given in exercise 1. In some cases, however, you will need to make your own tables of values.

a. Draw the graph of $y = x + 3$. Choose the horizontal axis as the x-axis.

b. Draw the graphs of the formulas listed below on the same set of axes used in exercise 4a. Extend each graph sufficiently far so that it intersects the y-axis. Label each graph by writing its formula along the graph.

$y = x - 3.$
$y = x + 2.$
$y = x - 2.$

c. Study the graphs that you drew for exercises 4a and 4b. Try to draw the graphs of the formulas listed below without making a table of values. Use the same set of axes that you used in exercises 4a and 4b.

$y = x + 1.$
$y = x - 1.$
$y = x.$

d. At what point would you expect the graph of $y = x + 100$ to intersect the y-axis? How do you think the steepness of this graph will compare with the steepness of each of the graphs that you drew in exercises 4a, 4b, and 4c?

e. What would you expect the graph of $y = x - 100$ to look like? Where would you expect it to intersect the y-axis?

5. In this exercise you are asked to draw the graphs of several formulas. Tables of values for some of the formulas are given in exercise 1.

a. Draw the graph of the formula $y = 3x$.

b. Draw the graphs of the formulas listed below. Use the same set of axes that you used in exercise 5a.

$$y = \frac{1}{3}x.$$

$$y = 2x.$$
$$y = \frac{1}{2}x.$$
$$y = x.$$

c. What would you expect the graph of $y = 100x$ to look like? What would you expect the graph of $y = \frac{1}{100}x$ to look like?

6. a. Write a formula which is different from those given in the exercises of this section and which has a graph that rises to the right.
   b. Write a formula which has a graph that falls to the right.

6 The Faulty Thermometer

Mr. Sweet gave thermometers to five students and asked each to record the temperature for each of six different substances. The temperatures recorded by the students are listed in the following chart.

SUBSTANCE	TEMPERATURES RECORDED				
	Carol	Patty	John	Jean	David
Air outdoors	77°	76.5°	25°	77.2°	77.1°
Air in the classroom	68°	67.6°	20°	68.2°	68.1°
Air in the freezer	5°	4.6°	−15°	5.2°	5.1°
Air in the gymnasium	59°	58.6°	15°	59.2°	59.1°
Drinking water	50°	49.6°	10°	50.2°	50.1°
Boiling water	212°	211.6°	100°	212.2°	212.1°

Looking at the above chart, Mr. Sweet's students made these observations:

"The differences in the temperatures recorded by Carol, Patty, Jean, and David for a particular substance are small and are probably due to errors in reading the thermometers."

"The temperatures that John recorded are way off! His thermometer must have been faulty."

Mr. Sweet suggested that the class examine the data before declaring John's thermometer faulty. Since Carol used only whole numbers in recording temperatures, the class decided to make a table of values to compare the temperatures recorded by Carol and John.

	Air Outdoors	Air in the Classroom	Air in the Freezer	Air in the Gymnasium	Drinking Water	Boiling Water
Temperatures recorded by Carol	77°	68°	5°	59°	50°	212°
Temperatures recorded by John	25°	20°	−15°	15°	10°	100°

Class Discussion 6

1. Assume that each *column* in the above table corresponds to an ordered pair of numbers and plot the points associated with these ordered pairs. Choose the horizontal axis for the temperatures recorded by Carol, and use the vertical axis for the temperatures recorded by John.
2. Do the points that you plotted suggest a pattern? Draw the simplest line connecting the points in order from left to right. Extend the line as far as you can on the sheet of paper you are using.
3. Use your graph to determine the reading on John's thermometer if the reading on Carol's thermometer is 32°. If it is 78°. If it is $^{-}10^{\circ}$.
4. What is the reading on Carol's thermometer if the reading on John's thermometer is 10°; 78°; $^{-}30^{\circ}$?

But Mr. Sweet did not give John a faulty thermometer. He gave him a perfectly good thermometer, but one that had a different scale than the thermometers he gave the other students. The thermometer that Mr. Sweet gave John had a centigrade scale, and the thermometers that he gave Carol and the other students had Fahrenheit scales. Thermometers with the Fahrenheit scale are widely used in homes and for many everyday purposes. Scientists make considerable use of thermometers with the centigrade scale, but they sometimes use thermometers having the Fahrenheit scale.

A convenient way to indicate that a temperature reading has been obtained with a centigrade scale is to write "C." after the reading. Similarly, to indicate that a reading has been obtained with a Fahrenheit scale we write "F." after the reading. For example, since John obtained an outdoors temperature reading of 25° by using a centigrade scale, we record his reading by writing 25°C. On the other hand, since Carol used a Fahrenheit scale, we record her reading by writing 77°F.

The formula $F = \frac{9}{5}C + 32$ may be used to convert a centigrade reading to the corresponding Fahrenheit reading. In this formula C is a variable for the number of degrees centigrade, and F is a variable for the number of degrees Fahrenheit. Let us use this formula to convert John's centigrade reading of 20°C. to Fahrenheit.

$$F = \frac{9}{5}C + 32.$$
$$F = \frac{9}{5}(20) + 32.$$
$$F = 36 + 32.$$
$$F = 68.$$

Therefore, the reading on the Fahrenheit scale that corresponds to 20°C. is 68°F. Compare this answer with Carol's reading shown in the second column of the table.

Exercises—6

1. a. During a three-day hot spell in Mexico City the high temperatures for the three days were 35°C., 31°C., and 38°C. Use the graph that you made in exercises 1 and 2 of Class Discussion 6 to determine the corresponding readings on a Fahrenheit scale.
   b. Now use the formula $F = \frac{9}{5}C + 32$ to convert each of the temperature readings given in exercise 1a to Fahrenheit.
   c. Which way of converting a temperature reading from

centigrade to Fahrenheit is more accurate — estimating by using a graph, or computing with the aid of the formula $F = \frac{9}{5}C + 32$?

d. Disregarding small errors, which of the methods described in exercise 1c is faster?

2. Let us suppose that we wish to convert a temperature reading from Fahrenheit to centigrade. The formula $F = \frac{9}{5}C + 32$ can again be used, but the computation will be easier if we use the formula $C = \frac{5}{9}(F - 32)$. The two formulas are equivalent. What does this mean?

a. Let us use the formula $C = \frac{5}{9}(F - 32)$ to convert Carol's reading of 59°F. to centigrade. Copy and complete the steps in the procedure shown below.

$$C = \frac{5}{9}(F - 32).$$

$$C = \frac{5}{9}(59 - 32).$$

$$C = \frac{5}{9}(?).$$

$$C = ?$$

The subtraction inside the parentheses should be done first; the difference should then be multiplied by $\frac{5}{9}$.

b. What reading on the centigrade scale corresponds to 59°F.? Compare your answer with John's reading shown in the fourth column of the table.

3. Decide which formula you should use for each conversion called for below. In each case check the result you obtain by using the graph that you made in exercises 1 and 2 of Class Discussion 6.

a. 41°F. to centigrade
b. 60°C. to Fahrenheit
c. 176°F. to centigrade

4. In a study of ants it was found that an ant's speed of movement is related to the temperature of the air by the formula $t = 11s + 39$. In this formula s is a variable for the ant's speed measured in inches per minute, and t is a variable for the temperature of the air measured in degrees Fahrenheit.

a. Copy and complete the following table of values.

s	1	2	3	4	5	6
t	50					

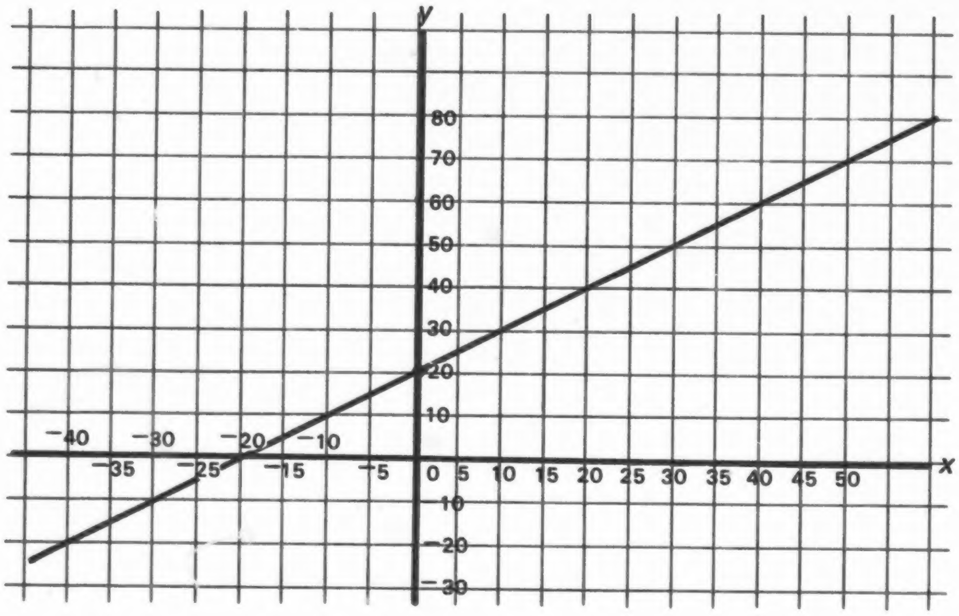
b. Make a graph of the formula, letting the horizontal axis be the s-axis.

c. Use your graph to find the missing values in the following table.

s		4½		5½	
t	55		77		39

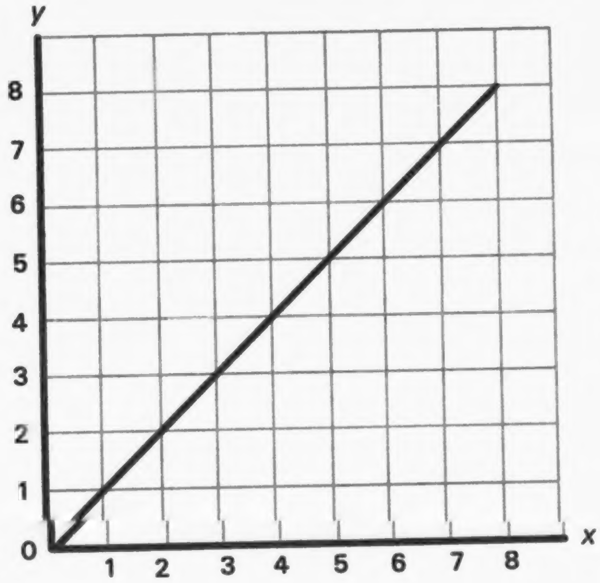
5. Copy and complete the table at the right by using the graph below.

x	30		20	0		−35		−10
y		70			0		−20	



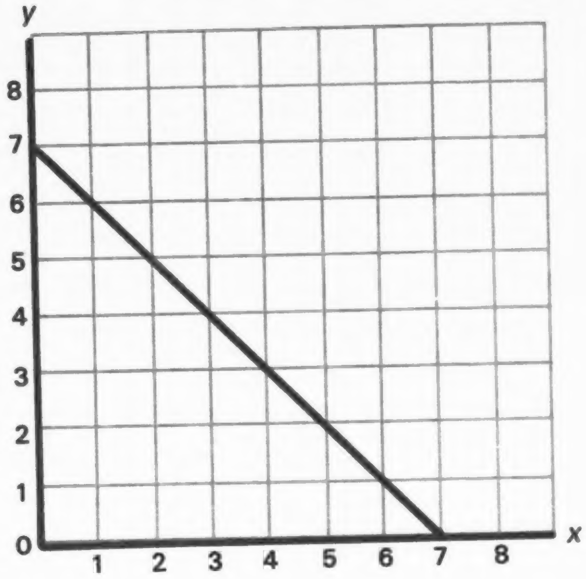
6. For each exercise examine the graph and determine which one of the formulas that appear to the right of the graph fits the graph. (You may find it helpful to make a brief table of values for each formula.)

a.



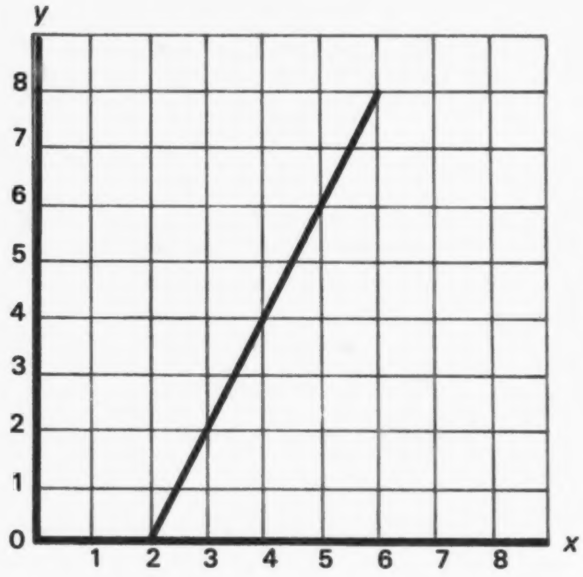
$y = x + 1.$
$2y = x.$
$y = x.$
$y = 2x.$

b.



$y = x.$
$y = x + 7.$
$y = 7 - x.$
$y = x - 7.$

c.



$y = 2x$.
$y = x - 1$.
$y = 2x - 2$.
$y = 2x - 4$.

7. Find a formula that fits the graph shown in exercise 5.

Summary—6

In an earlier section we learned how to make a table of values for a formula. By using the table of values we were able to obtain a graph of the formula.

In the work just completed we used graphs to help us find solutions of a formula. We also tried to find a formula that would fit a particular graph.

As a person continues his study of mathematics he learns to move easily from one way of expressing a relationship to another. Each way has its strengths and weaknesses. Ordinarily a table of values for a particular formula contains only a limited number of solutions of the formula. On the other hand, the number of solutions of the formula that can be read from its graph, or that can be computed by using the formula itself, is unlimited. Since a graph gives a picture of the relationship that exists among the variables in a formula, the graph of a formula is often more convenient to use than a table of values for the formula, or even the formula itself.

7 Stopping Distance of an Automobile

A car is traveling at the rate of 80 miles per hour. Suddenly the driver sees a sign that reads "Danger Ahead," and he applies the brakes in an effort to stop. How far do you think the car will travel before it comes to a complete stop? Less than 50 feet? Less than 100 feet? Less than 200 feet? Less than 350 feet? Make a prediction before you read any further.

Following is a formula for finding the average stopping distance of an automobile.

$$D = (.055) \times R \times R.$$

In this formula D is a variable for the distance in feet that the car travels after the driver applies the brakes, and R is a variable for the speed in miles per hour that the car is traveling when the brakes are first applied.

Class Discussion 7

1. Use the above formula to find the average stopping distance of a car traveling at the rate of 80 miles per hour. How does this result compare with the prediction that you made?

2. Another form of the formula given above is

$$D = .055R^2.$$

The 2 in the formula is an exponent. Note that the 2 is written in a raised position to the right of R. The symbol R^2 means $R \times R$.

a. What is the meaning of each of the following: a^2, 10^2, R^3?

b. What whole number equals 10^2? 20^2? 30^2? 40^2?

3. Use the formula $D = .055R^2$ to complete the table of values below.

R	0	10	20	30	40	50	60	70	90	100
D										

4. Interpreting each column of the table as an ordered pair of numbers as before, plot the point associated with each ordered pair. Let the horizontal axis be the R-axis.

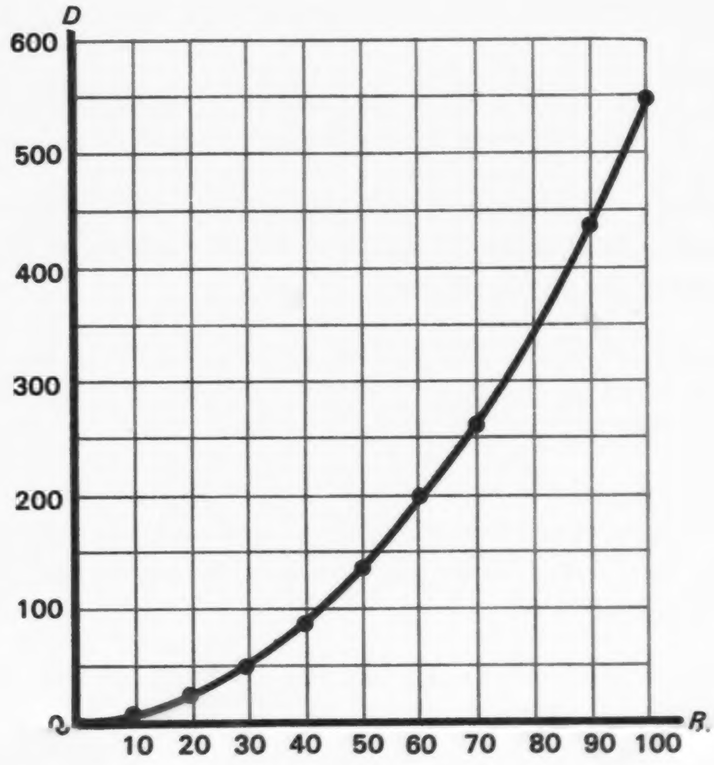
5. Do the points that you plotted suggest a pattern? If so, draw the graph suggested by the pattern. How does this graph differ from the graphs that you made before in this unit?

6. According to your graph, what is the approximate stopping distance if the brakes are applied when a car is traveling at the rate of 80 miles per hour? How does this compare with the result you obtained in exercise 1?

7. Use your graph to find the answer to each of the following questions.
   a. What is the approximate stopping distance if the brakes are applied when a car is traveling at the rate of 35 miles per hour? 55 miles per hour?
   b. If a car travels 300 feet before coming to a stop after the brakes are applied, how fast was the car traveling when the brakes were applied? If the car travels 100 feet, how fast was it traveling when the brakes were applied?
   c. There was an automobile accident in which two cars collided. An officer who arrived after the accident gave one of the drivers a ticket for speeding. How did he determine that the driver was speeding?

Summary—7

If you drew the graph in exercise 5 correctly, it should look like the one shown on the following page.



There is a major difference between this graph and the graphs you have drawn before in this unit. All the formulas you have drawn thus far had straight lines for their graphs. However, the formula $D = .055R^2$ has the graph shown above, and, as you can see, it is a curved line. It is usually possible to determine the shape of the graph of a formula by studying the formula. Perhaps you can discover a method of doing this while working with the exercises that follow.

Exercises—7

1. The formula $D = .055R^2$ introduced in the exercises in Class Discussion 7 can be used to find the average stopping distance of a car after the brakes are applied. Now let us consider the *reaction time* of the driver. The reaction time of the driver is the difference between the time that the driver gets the signal to stop

and the time that he applies the brakes. The following formula can be used to determine the distance a car travels during the reaction time:

$$D = 1.1R.$$

In this formula D represents distance in feet that the car travels during the reaction time, and R represents the rate of the car in miles per hour.

a. Use the formula to complete the table below.

R	0	10	20	30	50	100
D						

b. Let the horizontal axis be the R-axis and plot the points associated with the ordered pairs of numbers in the columns of the completed table. Draw the graph suggested by the pattern of the points that you plotted. This is the graph of the formula $D = 1.1R$.

c. How does the graph that you drew in exercise 1b compare with the graph of the formula $D = .055R^2$?

d. Use the graph of $D = 1.1R$ and the graph of $D = .055R^2$ to determine the total stopping distance of a car that is traveling at the rate of 47 miles per hour; 63 miles per hour; 95 miles per hour.

e. After the brakes are applied, a car travels 250 feet before stopping. How far did the car travel during the reaction time of the driver?

2. Galileo, an Italian astronomer, experimented with falling objects. He discovered that one can determine the distance an object falls in a given period of time if it starts from a position of rest. The formula that Galileo discovered is $D = 16T^2$, where T represents the time in seconds and D represents the distance in feet.

a. Copy and complete the table of values below.

T	0	1	2	3	4	5	6
D							

b. Draw a graph of the formula. Let the horizontal axis be the T-axis.

c. If it took 2½ seconds for a boulder to drop from the top of a cliff to the bottom of a ravine, how many feet did it drop? Use your graph to find the answer.

d. How many seconds will it take for a stone to drop to the bottom of a ravine that is 500 feet deep? Use your graph to find the answer.

3. Copy and complete the table of values for each formula.

a. $y = x + 5$.

x	0	1	2	3	4	5
y						

b. $y = 2(x + 1)$.

x	0	1	2	3	4	5
y						

c. $y = 12 - x$.

x	0	1	2	3	4	5
y						

d. $y = 2x^2$.

x	0	1	2	3	4	5
y						

e. $y = x^2 + 4$.

x	0	1	2	3	4	5
y						

4. a. If you were to draw the graph for each formula listed in exercise 3, which graphs would rise to the right? Try to answer this question without actually drawing the graphs.

b. Which of the formulas in exercise 3 have graphs that fall to the right?

c. Which formulas in exercise 3 have graphs that are straight lines?

5. A man has 36 feet of fencing with which to make a dog pen. He wishes to enclose a rectangular region with the greatest possible area, using the fencing that he has available. Use the formula $A = LW$, where A represents the number of square feet in the area, L represents the number of feet in the length, and W represents the number of feet in the width. Answer the following questions.

a. If the width of the fence is 1 foot, what is the length? What is the area?

b. If the width of the fence is 2 feet, what is the length? What is the area?

c. If the width of the fence is 3 feet, what is the area?

d. Copy and complete the table below.

W	1	2	3	5	6	7	9	12	14	16	17	18
L	17											
A	17											

e. The completed table in exercise 5d will contain ordered triples of numbers—that is, we can think of each column as an ordered triple of numbers. However, let us concern ourselves only with ordered pairs of numbers—that is, with ordered pairs in which the first number is the number of feet in the width of a rectangle and the second number is the number of square feet in the area of the rectangle. Choose the horizontal axis as the *W*-axis and plot the points associated with the ordered pairs of numbers described above. Without lifting your pencil, draw the graph suggested by the points that you plotted.

f. What is the area of the rectangular region with the greatest area that can be enclosed with 36 feet of fencing?

6. There is a definite pattern in each of the two lists of ordered pairs of numbers below. Try to discover what the pattern in each list is. Then list four more ordered pairs that fit the pattern. For each list, find a formula that would have the given ordered pairs (as well as those that you added) as solutions.

a.	**b.**
(0, 1)	(8, 3)
(1, 2)	(6, 4)
(2, 5)	(3, 8)
(3, 10)	(2, 12)
(4, 17)	(1, 24)

8 The Carefree Hostess

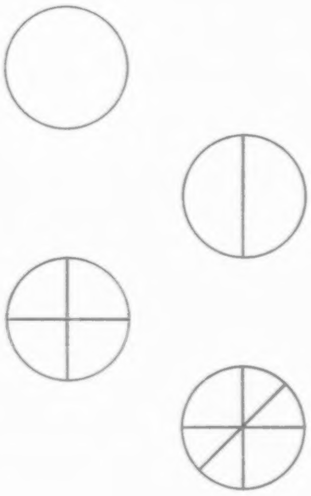
Most people enjoy the challenge of a good puzzle problem. To solve the problem described below, you need a good imagination.

Mrs. Adams is entertaining a number of people. She intends to serve each person a piece of pie. Unfortunately Mrs. Adams has only one pie. Moreover, she is quite carefree about serving her guests. She doesn't care about the size or shape of the pieces that she serves. In cutting the pie Mrs. Adams' only concern is that she obtain the maximum number of pieces. Each cut must be straight and must extend completely across the pie. Soon Mrs. Adams is so involved with the question of how many pieces can be obtained with different numbers of cuts that she forgets her guests. What Mrs. Adams wants to know is how many people she can serve in each case if she makes from one to eight cuts.

Before going on to the class-discussion exercises, make a guess as to what you think the answers might be.

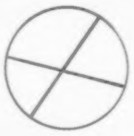
Class Discussion 8

1. Imagine that the pictures at the right represent pies.
   - **a.** If no guests arrive, then no cuts need to be made. If this is the case, how many pieces are there?
   - **b.** If one cut is made through the center of the pie, how many pieces are there?
   - **c.** If two cuts are made through the center of the pie, how many pieces are there?
   - **d.** If three cuts are made through the center of the pie, how many pieces are there?



- **e.** Suppose that Mrs. Adams continues to make cuts through the center of the pie in the manner suggested by the drawings. How many pieces will she have after making four cuts?
- **f.** Does the pie cutting described thus far suggest any number pattern to you? If so, what is it?
- **g.** If Mrs. Adams continues to cut the pie in the manner described above, how many pieces will she have after eight cuts?

2. All cuts described in the previous exercise were made through the center of the pie. According to the statement of the original problem, however, it is not necessary for the cuts to be made through the center. Suppose that two cuts are made as shown in the picture at the right. Is it possible to divide the pie into more than six pieces by making just one more cut? If so, explain how.



3. Now experiment with four cuts, assuming, as in exercise 2, that the cuts need not be made through the center. What is the greatest number of pieces that can be obtained with four cuts?

4. Copy and complete the following table.

Number of Cuts	0	1	2	3	4
Greatest Number of Pieces Possible	1				

5. a. Examine your table carefully. Without actually experimenting, try to predict the greatest number of pieces that can be obtained if five cuts are made.

b. Test your prediction by making a drawing that shows how the five cuts could be made.

6. Without doing any more experimenting, determine the greatest number of pieces that can be obtained with six cuts. With seven cuts. With eight cuts. Extend the table that you started in exercise 4 and record these results.

7. Choose axes and plot the ordered pairs of numbers indicated by the columns of your table. Does the set of dots that you made constitute the graph of some, or of all, of the solutions to Mrs. Adams' problem?

8. Without lifting your pencil, draw a line from left to right connecting the points that you plotted in exercise 7. Would you be justified in saying that this line is a graph of all the solutions of the problem? Why, or why not?
9. Does the graph of the solutions of the problem contain any points whose coordinates are fractions?
10. How does the graph that you made in exercise 7 differ from the graph of the formula for the stopping distance of an automobile?

Summary—8

Experimentation is very important in mathematics. In this lesson we used experimentation to help us determine the greatest number of pieces of pie that could be obtained with eight cuts. We began our experimentation with cases that were so simple that we could count the number of pieces. By examining our results we found that a definite number pattern seemed to exist. Then we extended this pattern and attempted to make certain predictions.

The graph of the solutions of the pie-cutting problem consists of a set of isolated dots. If a continuous line is drawn through these dots, the line will contain points that are not solutions of the problem. The coordinates of points that correspond to solutions of the problem must be whole numbers. Fractions cannot be used as coordinates, since neither the number of cuts nor the number of pieces can be represented by a fraction.

Exercises—8

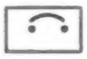
1. The formula below can be used to determine the greatest number of pieces of pie that can be obtained with a certain number of cuts. It is assumed that the cuts need not be made through the center.

$$P = \frac{C(C + 1)}{2} + 1.$$

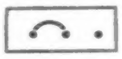
In this formula *P* represents the number of pieces of pie, and *C* represents the number of cuts. It is understood, of course, that only whole numbers can be used as replacements for *C*. Will *P* be a whole number for every permissible replacement of *C*? Give a reason for your answer.

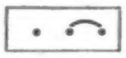
a. Use the formula to find *P* when *C* is 4.
b. Use the formula to find *P* when *C* is 8.
c. Use the formula to find *P* when *C* is 13.

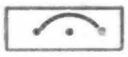
2. If a telephone switchboard has two lines coming into it, then exactly one connection can be made.



If a switchboard has three lines coming into it, then three different connections can be made.







a. Complete the following table.

Number of Lines	2	3	4	5	6	7
Number of Connections	1	3				

b. Make a graph of the ordered pairs of numbers indicated by the columns of the table. Assign a scale for the number of lines to the horizontal axis.
c. Does it make sense to connect the points that you plotted with a continuous line?

3. Experiments have shown that the number of chirps that a cricket makes increases as the temperature rises. The relationship between the number of chirps and the temperature is expressed by the following formula:

$$T = C + 37.$$

In this formula *T* represents the number of degrees Fahrenheit, and *C* represents the number of times a cricket chirps each fifteen seconds.

a. If a cricket chirps ten times in fifteen seconds, what is the temperature?

b. Use the formula to complete the table of values below.

C	0	5	10	15	20	25	30	35	40
T									

c. Make a graph that displays the information in the table.

d. Does it make sense to connect the points of this graph with a continuous line? Explain.

9 Height and Weight

The mathematics club at Kelly School decided to conduct an experiment. Each member was asked to collect information concerning the heights and weights of people in his neighborhood. The data that were gathered are recorded below as ordered pairs of numbers. The first number of each ordered pair represents the height in inches of some person, and the second number represents the weight in pounds of the same person.

(50, 67)
(48, 55)
(73, 169)
(74, 179)
(55, 70)
(60, 99)
(70, 149)
(51, 59)
(63, 119)
(67, 141)
(46, 50)
(56, 85)
(62, 135)
(56, 91)
(61, 119)
(54, 80)
(72, 180)
(61, 100)
(54, 86)
(64, 121)
(66, 136)
(65, 130)
(52, 63)
(64, 130)

Class Discussion 9

1. Plot the points associated with the ordered pairs of numbers given above.
2. Do the points that you plotted suggest any pattern?

3. Use your ruler to draw a straight line that passes through (or comes close to passing through) as many points as possible. A line drawn in this manner is called a *line of best fit.*

4. The formula $W = 5H - 190$ is sometimes used to express the relationship between a person's height and his weight. In this formula H represents the person's height in inches, and W represents his weight in pounds.

a. How well do you think this formula fits the line that you drew?
b. Predict your own weight by replacing H with the number of inches in your height.

5. **a.** According to the formula in exercise 4, what is the weight of a person who is 5 feet tall?
b. Use the line of best fit that you drew in exercise 3 to estimate the weight of a person who is 5 feet tall.
c. How do the results you obtained in exercises 5a and 5b compare?
d. Use the formula to compute a person's weight who is 50 inches tall and also find his weight by using the line of best fit. How do the results compare?
e. Make the same comparison as in exercise 5d for a height of 70 inches.

6. One student remarked that his three-year-old cousin is 38 inches tall.

a. According to the formula in exercise 4, how much should the cousin weigh?
b. How much should the cousin weigh according to the line of best fit?

7. Another student volunteered the information that his uncle, who is a professional football player, weighs 270 pounds.

a. What answer do you get when you use the formula to determine the uncle's height?
b. What answer do you get when you use the line of best fit to determine the uncle's height? Extend the line if necessary.

8. **a.** Can the formula in exercise 4 be used to obtain an accurate estimate of a person's weight if you know his height? Explain your answer.

b. Can the formula in exercise 4 be used to obtain an accurate estimate of a person's height if you know his weight? Explain your answer.

c. For what range of people's heights does the formula in exercise 4 give us reasonably accurate estimates of their weight?

Summary—9

One way to find a formula that expresses the relationship between the heights and weights of people in a group is to proceed in the following manner:

1. Begin by making a table in which related observations are recorded as ordered pairs of numbers. For the example mentioned, the ordered pairs of numbers would be of the form (number of inches in the height, number of pounds in the weight).
2. Plot on a grid the points associated with the ordered pairs of numbers in the table.
3. Draw a line of best fit.
4. Finally, examine the line of best fit and, by using trial-and-error methods, make some guesses as to what an appropriate formula might be. The ordered pairs of numbers associated with points on the line of best fit must be solutions of the formula.

In doing the exercises involving the formula $W = 5H - 190$, you undoubtedly discovered that this formula had to be used with caution. Within certain limits the formula worked reasonably well in computing either the height or the weight when one of these was given. However, for extreme heights (both very short and very tall) the computed weights were unrealistic. A similar statement can be made for computed heights. Thus, in making estimates by using graphs and formulas developed from observations, you need to determine within what range of values of the variables the graphs and the formulas give estimates that are reasonable.

Exercises—9

1. The formula below approximates the relationship between the age of a growing child and the number of hours of sleep he should have:

$$H = 17 - \frac{A}{2}.$$

In the formula, A represents the child's age in years, and H represents the number of hours of sleep he should have.

a. Copy the table below and complete it with the aid of the formula.

A	5	10	12	14	18
H					

b. Plot on a chart the points associated with the ordered pairs of numbers in the table. Let the horizontal axis be the A-axis. Draw a line of best fit for the points that you plotted. Extend the line as far as your paper permits.

c. Does the line that you drew rise to the right or fall to the right?

d. If a baby is one year old, how much sleep should he get? Compute your answer by using the formula and also estimate your answer by using the line of best fit.

e. According to the line of best fit, how much sleep should a person have if he is thirty-four years old? According to the formula, how much sleep should he have?

f. If we use the formula to compute the number of hours of sleep a person should have, do the answers we obtain make sense for all ages? For what range of ages does the formula give us reasonable answers?

2. Let x and y represent two quantities that are related. Some observed values of x and the corresponding values of y have been recorded in the table on the following page.

x	1	4	5	7	8	9
y	3	4	6	9	8	10

a. Plot the points associated with the ordered pairs of numbers indicated by the columns of the table. Let the horizontal axis be the x-axis.

b. Draw the line of best fit for the points that you plotted.

c. Which one of the formulas listed below best describes the relationship between x and y?

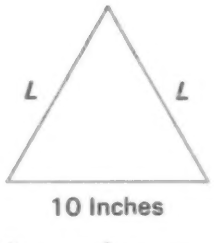
$$y = 2x.$$
$$y = x + 1.$$
$$y = x - 1.$$

d. Is the graph of every point that you plotted in exercise 2a on the line of best fit that you drew in exercise 2b?

3. Let us consider the set of triangles in which two sides of each triangle have the same length. Triangles of this kind are called *isosceles* triangles. The two sides that have the same length are referred to as the legs of the triangle, and the third side is called the base. In the picture at the right, L represents the length of each leg in inches. In the exercises that follow, we shall consider several different isosceles triangles, all having bases that are 10 inches long.

a. How many isosceles triangles are there with a base that is 10 inches long?

b. Suppose that the length L of each leg is 15 inches. What is the perimeter of the triangle? (The perimeter is the sum of the lengths of the three sides.)



c. If the length L of each leg is 20 inches, what is the perimeter of the triangle?

d. Which one of the following formulas can be used to find the perimeter P of an isosceles triangle that has a base of 10 inches?

$$P = 3L.$$
$$P = L + 10.$$
$$P = 2L + 10.$$
$$P = \frac{1}{2}L + 10.$$

e. Use the formula that you selected from the above list to complete the following table of values.

L	6	8	13	15	20	25
P	22					

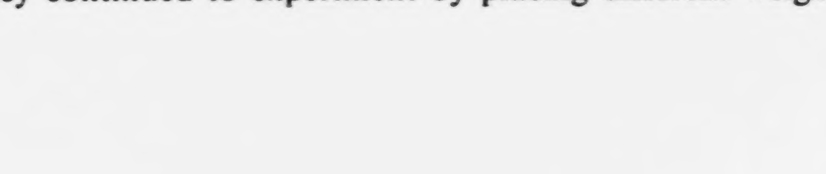
f. Use your completed table of values to help you draw a graph of the formula. Let the horizontal axis be the L-axis.

4. Consider once again the formula for the perimeter of an isosceles triangle with a base of 10 inches. Does it make sense to use this formula to find the perimeter of an isosceles triangle with a base of 10 inches if L is 5 inches? If L is 4 inches?

10 An Experiment in Science

The teacher asked Jim and Joe if they would be interested in investigating the properties of a lever. He explained to them that, since they had undoubtedly played on a teeter-totter, they were already familiar with one type of lever.

The boys decided to use a miniature teeter-totter for their lever and to learn by experimenting under what conditions the lever could be made to balance. To carry out their experiment, they obtained a thin board and a set of weights. They began their experiment by placing a weight of 2 ounces at a distance of 24 inches from the fulcrum (balancing point) on the left side of the lever as shown in the picture.



Jim and Joe suspected that the lever would balance if they placed a second weight of 2 ounces at a distance of 24 inches from the fulcrum on the right side of the lever. After confirming this fact they continued to experiment by placing different weights at

2

various distances from the fulcrum on the right side of the lever, while leaving the 2-ounce weight on the left side in its original position. The data they obtained are shown in the table below.

Weight in Ounces (*W*)	2	3	4	6	8	12
Distance from Fulcrum in Inches (*D*)	24	16	12	8	6	4

Class Discussion **10**

In the experiment described above the boys placed a weight of 2 ounces on the left side of the lever at a distance of 24 inches from the fulcrum. The questions in the exercises that follow involve the right side of the lever only and are based on the assumption that the weight of 2 ounces on the left side and its distance of 24 inches from the fulcrum remain fixed.

1. Study the table shown above. As the distance of the weight from the fulcrum decreases, what change takes place in the weight that is needed to balance the lever?
2. If the lever is in balance and the weight on the right side is doubled (for example, if it is changed from 4 ounces to 8 ounces), how must the distance of the weight from the fulcrum be changed if the lever is to remain in balance?
3. If the boys place a weight of 16 ounces on the right side, at what distance from the fulcrum must this weight be placed if the lever is to balance?
4. What weight placed 2 inches from the fulcrum on the right side will balance the lever?
5. Which one of the following formulas expresses the relationship between the quantities involved in balancing the lever that the boys have set up? In these formulas *W* represents weight in ounces, and *D* represents distance in inches.

$$D = 12W.$$
$$D = W + 22.$$
$$WD = 24.$$
$$WD = 48.$$

6. Use the correct formula from the list in exercise 5 to help you answer the following questions:
  - **a.** At what distance from the fulcrum on the right side should a weight of 5 ounces be placed if the lever is to be in balance?
  - **b.** At what distance from the fulcrum on the right side should a weight of 10 ounces be placed if the lever is to be in balance?
  - **c.** Are your answers to exercises 6a and 6b consistent with the answers you gave for exercise 2?

7. Again, use the correct formula from exercise 5 to help you answer the following questions:
  - **a.** What weight must be used to balance the lever if the weight is placed 5 inches from the fulcrum on the right side?
  - **b.** What weight must be used to balance the lever if the weight is placed 10 inches from the fulcrum on the right side?

8. Use the correct formula from exercise 5 to find D when W is replaced by 1.

9. Which one of the graphs shown below is a graph of the formula $WD = 48$?

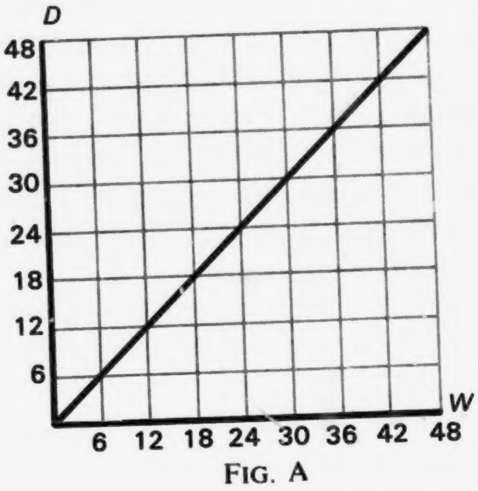


FIG. A

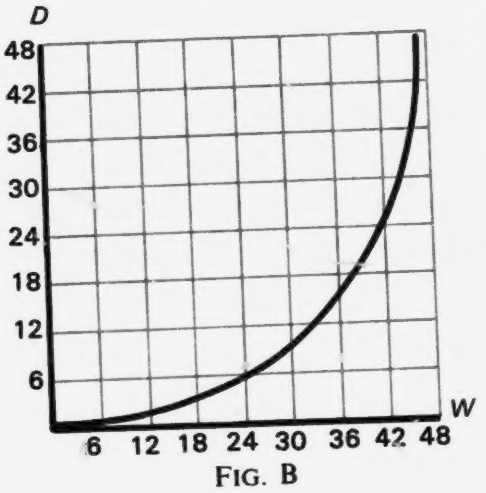


FIG. B

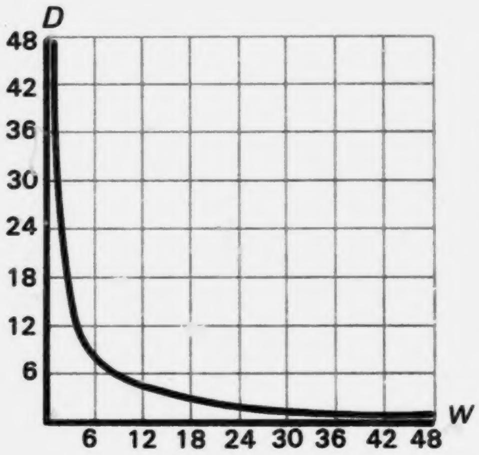


FIG. C

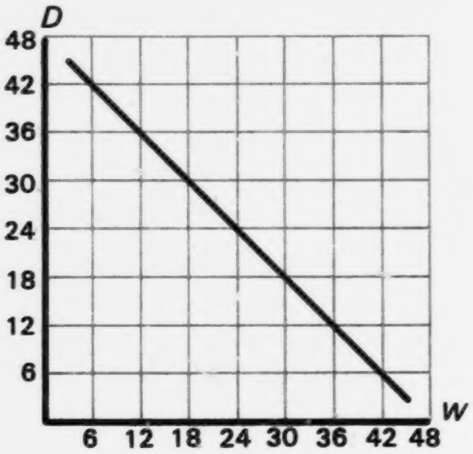


FIG. D

Summary—10

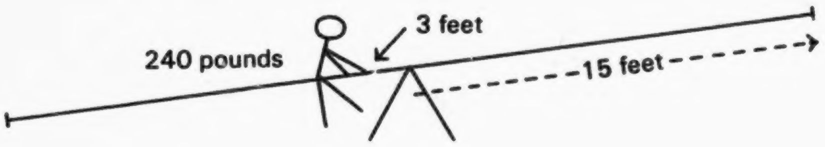
One of the first formulas presented in this chapter was $C = 30G$. Recall that this formula was used to find the cost of gasoline which sold for 30 cents per gallon. This formula has an interesting property. If the number of gallons of gasoline is doubled, so is the cost. For example, if the number of gallons of gasoline is increased from 2 to 4, then the cost is increased from 60 cents to 120 cents. In the same way, if the number of gallons of gasoline is multiplied by three, then the cost is also multiplied by three; and in general, if the number of gallons sold is multiplied by a certain number, then the cost is multiplied by the same number. Accordingly, we say that the cost *varies directly* as the number of gallons sold. A relationship expressed by a formula that has this property is called *direct variation*. Several of the formulas we have studied in this unit express direct variation. The following are typical:

$$D = 6t.$$
$$C = 3N.$$
$$D = 1.1R.$$

The formula $WD = 48$, which was introduced in the discussion exercises just completed, expresses a different kind of variation than do the formulas listed above. Recall that when the weight was doubled, the distance was halved, and that when the distance was doubled, the weight was halved. Similarly, when the weight was multiplied by three, the distance was divided by three. When two quantities are related in this manner, we say that one quantity *varies inversely* as the other. A formula that expresses such a relationship is said to express an *inverse variation*.

Exercises—10

1. The Owens children insisted that their father play with them on their teeter-totter. Mr. Owens, who weighs 240 pounds, sat on one side of the teeter-totter at a distance of 3 feet from the fulcrum, as shown in the picture.



Dick	180 pounds
Lloyd	120 pounds
Frances	100 pounds
Jack	60 pounds
Mary	50 pounds
Rickey	40 pounds

The weights of the Owens children are given in the chart at the right. Each child wishes to teeter alone with Mr. Owens. If Mr. Owens "stays put" where he sat down, how far from the fulcrum must each child sit in order to balance the teeter-totter? Since Mr. Owens' weight is 240 pounds and he sits 3 feet from the fulcrum, we can use the formula $WD = 720$ to determine how far from the fulcrum each child must sit in order to balance the teeter-totter. Explain why this formula is appropriate.

a. How far from the fulcrum must Lloyd sit in order to balance the teeter-totter? How far from the fulcrum must Dick sit? How far from the fulcrum must Frances sit?

b. How far from the fulcrum must Jack sit in order to balance the teeter-totter? How far from the fulcrum must Mary sit?

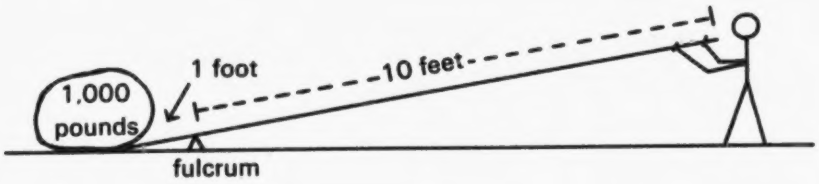
c. If the fulcrum remains at the point shown in the picture, is it possible for Rickey to balance the teeter-totter?

d. Suppose that Mr. Owens moves back 3 feet. Where must Dick sit now in order to balance the teeter-totter?

e. Suppose that a teeter-totter 40 feet long is used and that the fulcrum is placed at the center. At what distance from the fulcrum must Mr. Owens sit in order to balance the teeter-totter if Rickey sits on the extreme end of the opposite side?

2. Using a lever to lift a weight involves the same principle of levers as does balancing a teeter-totter.

a. A steel bar 11 feet long is used as a lever to raise a rock weighing 1,000 pounds. If the fulcrum is placed 1 foot from the rock, how many pounds of force must be exerted in a downward direction at the other end of the bar in order to raise the rock?



b. If the fulcrum is moved so that it is 2 feet from the rock, how many pounds of force will be needed to raise the rock?

3. The distance from Grangeville to Oceanside is 80 miles.

a. If a car travels at the rate of 40 miles per hour, how long will it take the car to make the trip from Grangeville to Oceanside?

b. If a car travels at the rate of 30 miles per hour, how long will it take the car to make the trip?

c. How long will it take a cyclist to make the trip if he travels at the rate of 20 miles per hour?

d. A formula that can be associated with exercises 3a, 3b, and 3c is $RT = 80$. In this formula R represents the rate in miles per hour, and T represents the time in hours. Use this formula to complete the following table of values.

R	80	40	20	16	10	5	4	2	1
T									

e. Draw a graph of the formula $RT = 80$. Let the horizontal axis be the R-axis.

4. The formula for finding the area of a rectangle is $A = LW$. Suppose that the area of a rectangle is 36 square feet.

a. List six possible combinations of length and width that will give an area of 36 square feet.

b. Use the formula $36 = LW$ to complete the following tables.

L	36	72	144	288
W				

L	1	1/2	1/4	1/8
W				

c. Draw a graph of the formula $36 = LW$. Let the horizontal axis be the L-axis.

d. If you extend the graph sufficiently far, will the graph intersect the L-axis? Will it intersect the W-axis? Explain your answers.

5. Consider again the previous exercise in which the area of the rectangle is given as 36 square feet.

a. For what length and width will the rectangle have the least perimeter?

b. Is it possible to find a length and width for which the perimeter will be greatest? Explain your answer.

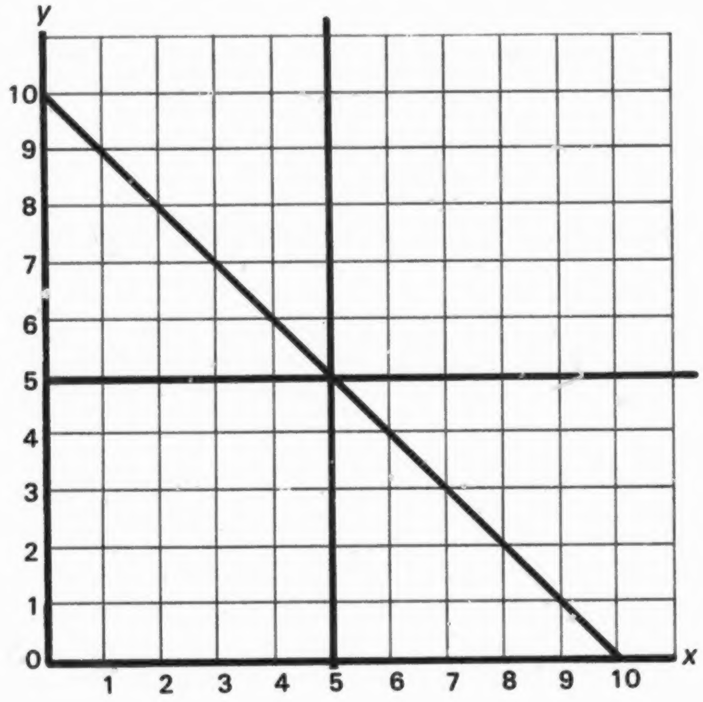
Review Exercises

1. Plot the points associated with the ordered pairs of numbers given below.

(4, 3) (2, 9) (0, 5) (3, 4) (6, 0) (0, 0)

2. Make use of the graph below to complete the table of values.

x	0	2	4	6	8	10
y						



3. List three ordered pairs of numbers in which the second number in each ordered pair is one less than the first number.

4. Plot the points associated with each of the following sets of ordered pairs. Try to find a pattern for each set and list five additional ordered pairs that fit the pattern.

- **a.** (4, 3), (5, 8), (6, 13), (7, 18)
- **b.** (2, 8), (4, 7), (6, 6), (8, 5)
- **c.** (6, 3), (5, 2), (4, 1), (3, 0)
- **d.** (8, 2), (6, 3), (4, 4), (2, 5)

5. Each table of values below contains ordered pairs of numbers which are solutions of one of the formulas listed at the right. Match each table of values with the appropriate formula.

A

x	2	3	4	6	12
y	12	8	6	4	2

B

x	0	2	4	6	8
y	1	7	13	19	25

C

x	0	1	3	5	7	9
y	0	1	9	25	49	81

a. $y = x^2$.
b. $y = 3x + 1$.
c. $y = x + 10$.
d. $xy = 24$.

6. Make a table of values for the formula $M = 2N + 1$ and draw the graph of the formula. Let the horizontal axis be the N-axis.

7. In each of the following, complete the table of values for the given formula.

a. $y = 7x$.

x	1	2	3	4	5	6
y						

b. $y = \frac{x}{3}$.

x	3	6	9	12	15	18
y						

c. $y = \frac{24}{x}$.

x	1	2	3	4	6	8	12
y							

d. $y = x^2$.

x	0	1	2	3	4	5	6
y							

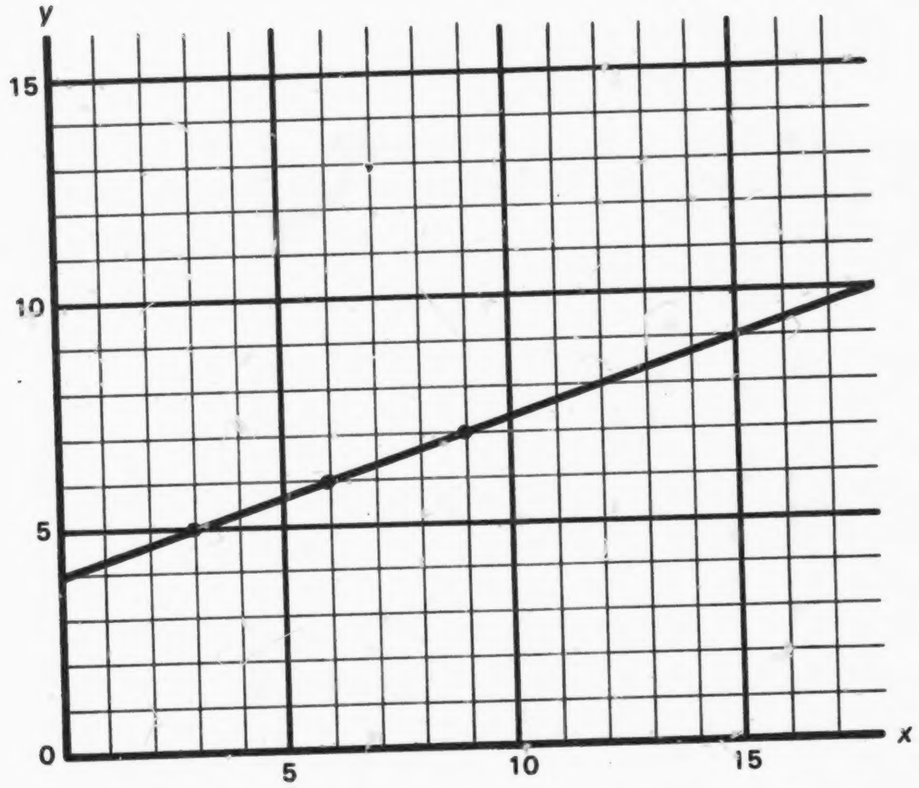
e. $y = 10 - x$.

x	2	4	6	8	10
y					

8. a. If a graph were drawn for each formula of exercise 7, which of the graphs would be curved lines?
b. Which of the graphs would fall to the right?
c. Which of the graphs would pass through the origin?

9. Which one of the formulas listed at the right has the graph shown below?

$y = x + 4.$
$y = 3x + 4.$
$y = \frac{1}{3}x + 4.$



10. Complete a table of values like the one at the right for each of the formulas listed below.

x	2	9	4	10
y				

a. $y = 2x + 3.$
b. $y = 3(x + 9).$
c. $y = 7(x - 2).$
d. $y = 5x^2.$
e. $y = x^2 + 6.$
f. $y = \frac{x}{3} + 11.$
g. $y = \frac{3}{4}x - 1.$

2

Properties of Operations with Numbers

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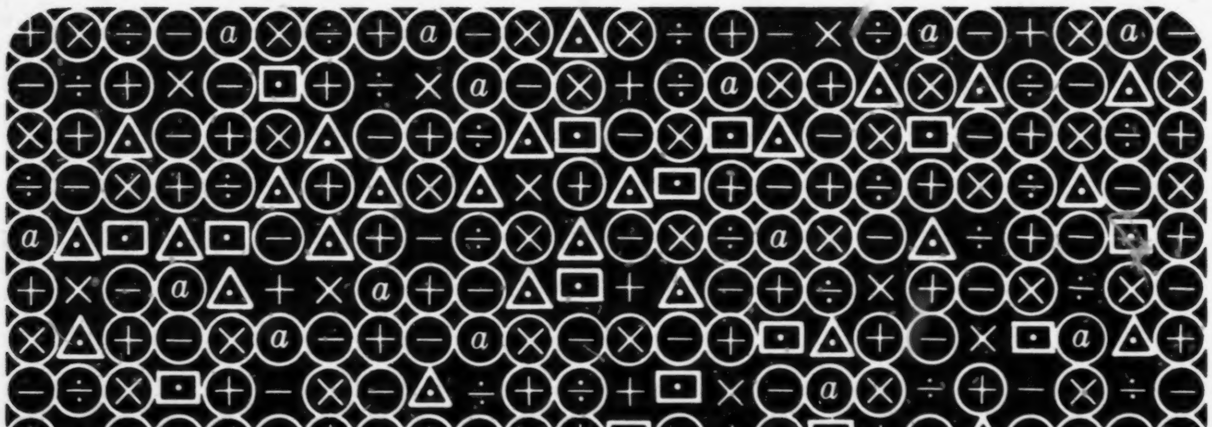

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UNIT TWO OF

Experiences in Mathematical Discovery

Properties of Operations with Numbers

NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

1201 Sixteenth Street, N.W., Washington, D.C. 20036

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Preface

"Experiences in Mathematical Discovery" is a series of ten self-contained units, each of which is designed for use by students of ninth-grade general mathematics. These booklets are the culmination of work undertaken as part of the General Mathematics Writing Project of the National Council of Teachers of Mathematics (NCTM).

The titles in the series are as follows:

Unit 1: *Formulas, Graphs, and Patterns*

Unit 2: *Properties of Operations with Numbers*

Unit 3: *Mathematical Sentences*

Unit 4: *Geometry*

Unit 5: *Arrangements and Selections*

Unit 6: *Mathematical Thinking*

Unit 7: *Rational Numbers*

Unit 8: *Ratios, Proportions, and Percent*

Unit 9: *Measurement*

Unit 10: *Positive and Negative Numbers*

This project is experimental. Teachers may use as many units as suit their purposes. Authors are encouraged to develop similar approaches to the topics treated here, and to other topics, since the aim of the NCTM in making these units available is to stimu-

late the development of special materials that can be effectively used with students of general mathematics.

Preliminary versions of the units were produced by a writing team that met at the University of Oregon during the summer of 1963. The units were subsequently tried out in ninth-grade general mathematics classes throughout the United States.

Oscar F. Schaaf, of the University of Oregon, was director of the 1963 summer writing team that produced the preliminary materials. The work of planning the content of the various units was undertaken by Thomas J. Hill, Oklahoma City Public Schools, Oklahoma City, Oklahoma; Paul S. Jorgensen, Carleton College, Northfield, Minnesota; Kenneth P. Kidd, University of Florida, Gainesville, Florida; and Max Peters, George W. Wingate High School, Brooklyn, New York.

The Advisory Committee for the General Mathematics Writing Project was composed of Emil J. Berger (*chairman*), St. Paul Public Schools, St. Paul, Minnesota; Irving Adler, North Bennington, Vermont; Stanley J. Bezuszka, S.J., Boston College, Chestnut Hill, Massachusetts; Eugene P. Smith, Wayne State University, Detroit, Michigan; and Max A. Sobel, Montclair State College, Upper Montclair, New Jersey.

The following people participated in the writing of either the preliminary or the revised versions of the various units:

Donald H. Firl, Rochester Public Schools, Rochester, Minnesota

Carroll E. Fogal, University of Florida, Gainesville, Florida

Edward F. Gottlieb, Madison High School, Portland, Oregon

Kenneth P. Kidd, University of Florida, Gainesville, Florida

Edna K. Lassar, Polytechnic Institute of Brooklyn, Brooklyn, New York

John F. LeBlanc, Racine Public Schools, Racine, Wisconsin

Scott D. McFadden, Spencer Butte Junior High School, Eugene, Oregon

Max Peters, George W. Wingate High School, Brooklyn, New York

Leonard Simon, Bureau of Curriculum Research, New York Public Schools, New York, New York

Irwin N. Sokol, Collinwood High School, Cleveland, Ohio

William H. Spragens, University of Louisville, Louisville, Kentucky

Mary E. Stine, Fairfax County Schools, Fairfax, Virginia

Arthur J. Wiebe, Pacific College, Fresno, California

John E. Yarnelle, Hanover College, Hanover, Indiana

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EMIL J. BERGER
Chairman, Advisory Committee
General Mathematics Writing Project

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Experiences in Mathematical Discovery

Properties of Operations with Numbers

1 Days of the Week

Today is Tuesday. What day of the week will it be ten days from today? How would you work out the correct answer to such a question? One approach might be to count on your fingers. Beginning with Wednesday, you might say, "Wednesday, Thursday, Friday, ... ," until you come to the day of the week that matches the tenth finger. Do you agree that this would be Friday?

So far so good. But suppose you were asked, "What day of the week will it be thirty days from today? Or fifty days from today?" Counting on your fingers would be a slow and tiresome way of finding answers to these questions. Can you think of a better way, using mathematics?

1966 AUGUST 1966

S	M	T	W	T	F	S
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31			

Class Discussion 1

1. Let us number the days of the week, beginning with 0 (zero) for Sunday, 1 for Monday, 2 for Tuesday, etc.
   - **a.** What number corresponds to Wednesday? To Friday?
   - **b.** What is the greatest number that is needed to number the days of the week?
   - **c.** To what day of the week does 6 correspond?
2. The number for Tuesday is 2. Consider again the question, What day of the week will it be ten days from Tuesday? If we add 2 and 10, we get 12. The answer to the question is Friday. And the number for Friday is 5. What is the difference of 12 and 5?
3. If today is Thursday, what day of the week will it be twenty days from today? Find the answer by finger counting or by using the calendar on the previous page.
4. Wednesday is the answer to the question in the last exercise, and the number that corresponds to Wednesday is 3. Now add 4 (the number for Thursday) and 20.
5. The number you obtain if you add 4 and 20 is 24. What is the difference of 24 and 3?
6. The difference of 24 and 3 is equal to three times what number?
7. Note that if we subtract three times seven from 24, the result is 3. How is the number that we subtracted from 24 related to the number of days in a week?
8. Can you suggest a general method for answering the question, What day of the week will it be a certain number of days from today?
9. What day of the week will it be thirty days from Monday? $1 + 30 = 31$. What multiple of seven should be subtracted from 31 to obtain a number that corresponds to a day of the week—that is, to a number that is less than seven?

10. If 28 is subtracted from 31, the resulting number is 3. What day of the week corresponds to 3?

Exercises—1

1. Make a chart showing the numbers corresponding to the days of the week, beginning with zero for Sunday.
2. Use the method suggested in Class Discussion 1 to determine on what day of the week each of the following days will fall.
   - a. Thirty days after Tuesday
   - b. Twenty-five days after Friday
   - c. Forty-two days after Wednesday
   - d. Fifty days after Saturday
   - e. Sixty-five days after Sunday
   - f. Seventy-five days after Thursday
3. The month of March has 31 days. If March 10 falls on Wednesday, on what day of the week does April 10 fall?
4. This year John's birthday falls on Saturday. His brother's birthday is sixty days later. On what day of the week will his brother's birthday fall?
5. If Christmas falls on Wednesday in a certain year, on what day of the week will it fall a year later? (Assume that there are 365 days in a year.)

2 Circle Arithmetic

In doing each exercise in the previous section you should have added two numbers and then subtracted the greatest multiple of 7 that is not greater than the sum of the two numbers.

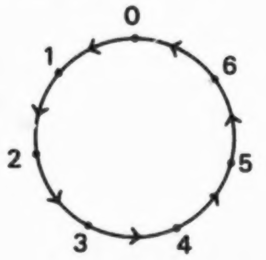
For example, to find the answer in exercise 2a, you should have found the sum $2 + 30 = 32$ and then subtracted 4×7 from 32. That is, you should have obtained 32 − 28 = 4 as the result. The procedure involving addition of 2 and 30 and subtraction of 4 × 7 from the result may be represented symbolically by writing

$$2 \oplus 30 = 4.$$

Note that we have used the "plus" symbol with a circle around it ($\oplus$) to indicate the procedure described above. We shall refer to this procedure as the operation of *circle addition*. The reason for the name will appear shortly. Explain why we can say that $6 \oplus 5 = 4$.

Class Discussion 2

1. The circle-addition operation described above can be illustrated by using a circle on which equally spaced points are associated with the numbers 0, 1, 2, 3, 4, 5, and 6 as shown below. We shall refer to this circle as a number circle.



To find $3 \oplus 6$, begin at the point labeled 0, move 3 spaces in the direction indicated by the arrows, and, after that, move 6 more spaces in the same direction. At what point do you arrive?

2. Begin again at the point labeled 0, travel 5 spaces in the direction of the arrows, and then travel 4 more spaces in the direction of the arrows. Where do you arrive?

3. Describe the result obtained in exercise 2 by writing a mathematical sentence that includes the symbol $\oplus$.

4. Find the result of the circle addition in each of the following:

a. $4 \oplus 6$
b. $6 \oplus 6$
c. $4 \oplus 5$
d. $2 \oplus 2$

5. The result of the circle addition in exercise 4d turns out to be the same as the result of ordinary addition. When will this happen?

6. Begin at the point labeled 0 and move 4 spaces in the direction of the arrows 5 times. Where do you arrive?

7. Describe the result of the operation in exercise 6 by writing a mathematical sentence that includes the "times" symbol with a circle around it $\otimes$. Do you see that the sentence could be written as $5 \otimes 4 = 6$? Let us call the operation symbolized by $\otimes$ *circle multiplication*.

8. Consider the example $5 \otimes 3$. The result of the circle multiplication can be obtained by starting at 0 on the circle and moving 3 spaces 5 times in the direction of the arrows. Do you see a way of getting the same result using ideas from the previous section?

9. If you find the product 5×3 by multiplying as in ordinary arithmetic and then subtract the greatest multiple of 7 that is not greater than this product, what result do you obtain?

Exercises—2

1. In each of the following find the result of the indicated circle operation:

a. $6 \oplus 5$
b. $3 \oplus 4$
c. $6 \otimes 5$
d. $3 \otimes 4$
e. $4 \oplus 6$
f. $5 \otimes 6$
g. $3 \oplus 2$
h. $3 \oplus 5$
i. $5 \oplus 3$
j. $5 \otimes 5$

2. Construct a table for circle addition. The chart below shows how to make such a table. The results of the following circle additions have already been entered:

$$3 \oplus 4 = 0.$$
$$4 \oplus 5 = 2.$$
$$5 \oplus 3 = 1.$$

⊕	0	1	2	3	4	5	6
0							
1							
2							
3					0		
4						2	
5				1			
6							

3. Construct a table for circle multiplication like the one started below.

⊗	0	1	2	3	4	5	6
0			0				
1							6
2							
3					5		
4			1				
5						4	
6							

Summary—2

In this section we introduced a mathematical system that we called circle arithmetic. The system we introduced involves the set of numbers $\{0, 1, 2, 3, 4, 5, 6\}$ and two operations—circle addition ⊕ and circle multiplication ⊗.

There are other systems of circle arithmetic, each involving the same kind of circle operations but a different set of numbers. Because all explorations of circle arithmetic undertaken in this unit deal only with the system that involves the set of numbers {0, 1, 2, 3, 4, 5, 6}, from now on we shall refer to the system that we have introduced as *the* system of circle arithmetic.

Both circle addition and circle multiplication are *binary* operations. (The prefix "bi-" denotes two.) What this means is that circle addition is defined for two numbers, and so is circle multiplication. The circle-addition and circle-multiplication tables you constructed should make this clear.

In the work that follows we shall refer to the result of circle addition $\oplus$ as a *sum*, and to the result of circle multiplication $\otimes$ as a *product*. Both sums and products in circle arithmetic can be found by moving around a number circle or by subtracting multiples of seven.

3 The Circle Is Closed

We have been exploring how to add and multiply numbers using circle-arithmetic operations. You learned that you can find the sum or the product of any two numbers in the set {0, 1, 2, 3, 4, 5, 6} by subtracting multiples of 7, or by moving around a circle.

Now that we have made a start, let's explore some other ideas in this new kind of arithmetic. We will be especially interested in the properties of the operations of circle addition and circle multiplication.

Class Discussion 3

1. Consider again the set of numbers {0, 1, 2, 3, 4, 5, 6}. Recall that the numbers in this set are associated with equally spaced points on a circle. Add any two numbers in this set, using circle addition. Will the result always be a number in the set?

2. If you multiply any two numbers in the set {0, 1, 2, 3, 4, 5, 6}, using circle multiplication, will you always get a number in the set?

3. Check your results in exercises 1 and 2 by referring to the tables you constructed in the previous section. Is every answer that you obtained in exercises 1 and 2 a number in the set {0, 1, 2, 3, 4, 5, 6}?

4. But suppose someone says, "5 x 6 = 30, and thirty is not in the original set." How would you answer this objection?

Your answers to the questions in exercises 1, 2, and 3 should suggest that if you add or multiply any two numbers in the set {0, 1, 2, 3, 4, 5, 6}, using circle arithmetic operations, then the resulting sum or product is *always* a number in the set. The last statement is really a statement of two important properties. These are expressed below.

Closure property of circle addition. The set of numbers {0, 1, 2, 3, 4, 5, 6} is closed with respect to circle addition.

Closure property of circle multiplication. The set of numbers {0, 1, 2, 3, 4, 5, 6} is closed with respect to circle multiplication.

5. Now let us consider the set of numbers {1, 2, 3, 4, 5}. Is this set closed with respect to ordinary addition?

6. In ordinary arithmetic we know that 2 + 2 = 4, and that 2 + 3 = 5. Both 4 and 5 are in the set {1, 2, 3, 4, 5}. But we also know that 3 + 4 = 7, and that 7 is not in the set {1, 2, 3, 4, 5}. When we find the sum of certain pairs of numbers in the set {1, 2, 3, 4, 5}, using ordinary addition, we do get a number in the set. On the other hand, for some pairs of numbers in the set, the sum is not in this set. For a set to be closed with respect to an operation, the result of applying the operation to *any* two numbers in the set must be a number in the set. Is the set {1, 2, 3, 4, 5} closed with respect to ordinary multiplication?

7. Give an example of two numbers in the set {1, 2, 3, 4, 5} whose product obtained by ordinary multiplication is in the set.

8. Find two numbers in the set {1, 2, 3, 4, 5} whose product obtained by ordinary multiplication is *not* in the set.

Exercises—3

1. Consider the set of even numbers {0, 2, 4, 6, 8, 10, 12,... }. (The three dots show that the numbers in the set continue without end.) Is this set closed with respect to ordinary addition?

2. Is the set of even numbers closed with respect to ordinary multiplication?

3. Is the set of odd numbers {1, 3, 5, 7, 9, 11, 13, ... } closed with respect to ordinary multiplication?

4. Is the set of odd numbers closed with respect to ordinary addition?

5. Consider now the set of numbers {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ... }. This set is called the set of *whole numbers*. Is the set of whole numbers closed with respect to ordinary multiplication?

6. Counting by fives, we obtain the set of numbers {5, 10, 15, 20, 25, 30, ... }. Is this set closed with respect to ordinary multiplication?

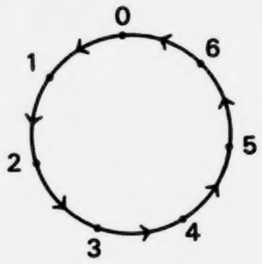
7. Is the set of multiples of 5 closed with respect to ordinary addition?

8. Consider the set of multiples of 3; that is, consider the set {3, 6, 9, 12, 15, ... }. In ordinary arithmetic, is the sum of any two multiples of three another multiple of three? Is the product of any two multiples of three a multiple of three?

9. Is the set of multiples of 3 closed with respect to ordinary addition? With respect to ordinary multiplication?

4 Turn About Is Fair Play

Let's take another look at the number circle that we introduced for circle arithmetic.



Class Discussion 4

1. In describing circle addition we said that to find the sum $4 \oplus 6$ we could begin at point 0, move 4 spaces in the direction of the arrows, and then move 6 more spaces in the direction of the arrows. If we do this, we arrive at point 3. Hence, $4 \oplus 6 = 3$. Now suppose we ask, What is $6 \oplus 4$? This time we would first move 6 spaces in the direction of the arrows, then move 4 more spaces in the direction of the arrows. Do we arrive at the same point as before? In other words, does $4 \oplus 6 = 6 \oplus 4$?

2. Find the sum $3 \oplus 5$. Then find the sum $5 \oplus 3$. Are the two results the same?

3. The results of exercises 1 and 2 indicate that in circle arithmetic the order in which two numbers are added does not matter. If you check the circle-addition table, you can see that this is true for any two numbers in the set $\{0, 1, 2, 3, 4, 5, 6\}$. Hence we say that circle addition has the *commutative property*. Do you think that circle multiplication also has the commutative property?

4. To support your answer to the question in exercise 3, recall that to find the product $5 \otimes 3$ we started at 0 on the number circle

and moved 3 spaces 5 times in the direction of the arrows. (See exercise 8 of Class Discussion 2.) How would you find the product $3 \otimes 5$? Do you end up at the same point in each case?

5. If two things can be done one after the other, and if the order in which the two things are done can be reversed without changing the result, we say that the operation of combining the actions is commutative. For example, if Dick puts on his left shoe first and then puts on his right shoe, the result is that both shoes are on. Of course, the same result is obtained if Dick puts on his right shoe first and then puts on his left shoe. Since the two results are the same, combining the actions in this case is commutative. For which of the following pairs of actions is the operation of combining the actions commutative?

a. Putting on a sock; putting on a shoe.
b. Walking 2 steps forward; walking 3 steps forward.
c. Taking the wrapper off a piece of chewing gum; chewing the gum.
d. Locking the car doors; getting into the car.
e. Earning 2 dollars one day; earning 4 dollars the next day.

Exercises—4

1. Let us consider a new operation $\boxdot$ described in the following way:

If a and b represent numbers, then

$$a \boxdot b = 2a + b.$$

We can also describe the operation $\boxdot$ by saying: Given two numbers, double the first number and add the result to the second number. For example, $5 \boxdot 3 = 10 + 3 = 13$. Find the result in each of the following:

a. $7 \boxdot 6$
b. $3 \boxdot 4$
c. $5 \boxdot 5$
d. $12 \boxdot 6$
e. $6 \boxdot 12$

2. Is the operation $\boxdot$ commutative?

3. Is ordinary addition of whole numbers commutative?

4. Is ordinary multiplication of whole numbers commutative?

5. Find the sum in each of the following, using ordinary arithmetic:

a.	**b.**	**c.**	**d.**
362	285	482	193
285	362	193	482

6. What do you observe about the answers to 5a and 5b? About the answers to 5c and 5d?

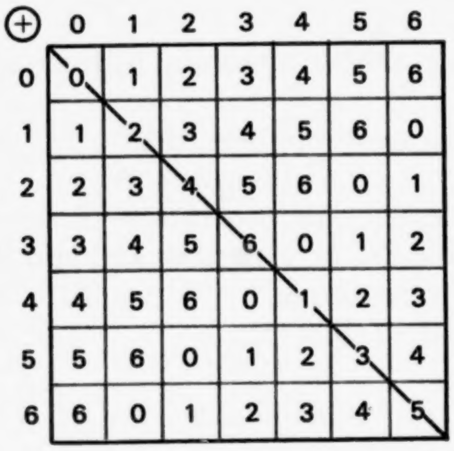
7. What property is illustrated by the examples in exercise 5?

8. Find the product in each of the following, using ordinary arithmetic:

a.	**b.**	**c.**	**d.**
38	45	96	72
72	96	45	38

9. Which pairs of examples have the same products? What property does this illustrate?

10. Let us return now to circle arithmetic. Bring out the circle-addition table that you were asked to produce in exercise 2 of Exercises—2. Draw a diagonal line in the table as shown below and fold your table along this line. Disregarding the entries in the diagonal, what do you observe about the pairs of entries that match? What property of circle arithmetic does this illustrate?



⊕	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

11. Carry out the same procedure as in exercise 10 with the circle-multiplication table that you were asked to produce in exercise 3 of Exercises—2.

12. Below is part of a circle-addition table. What shortcut can you use to find the answers to the circle-addition combinations that are missing?

⊕	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1		2	3	4	5	6	0
2			4	5	6	0	1
3				6	0	1	2
4					1	2	3
5						3	4
6							5

13. What shortcut can you use to find the answers to the circle-multiplication combinations that are missing in the table started below?

⊗	0	1	2	3	4	5	6
0	0						
1	0	1					
2	0	2	4				
3	0	3	6	2			
4	0	4	1	5	2		
5	0	5	3	1	6	4	
6	0	6	5	4	3	2	1

14. Suppose that some operation symbolized by $\triangle$ is commutative. If $a \triangle b = c$, what can you say about $b \triangle a$?

5 A Number Association

The circle-addition table in the system of circle arithmetic gives us a sum for every combination of two numbers in the set {0, 1, 2, 3, 4, 5, 6}. Similarly, the circle-multiplication table gives us a product for every combination of two numbers in this set. Since circle addition and circle multiplication are each defined for two numbers, we say that these operations are binary operations. (See the summary at the end of Section 2.) Ordinary addition and multiplication are also binary operations.

Now suppose that we wish to find the sum of three numbers in the system of circle arithmetic. For example, suppose we want to know what number is represented by $4 \oplus 5 \oplus 3$. Since circle addition is a binary operation, we cannot use the circle-addition table directly. However, we can find an answer by using a scheme that involves two steps.

Step 1: From the table, $4 \oplus 5 = 2$.
Step 2: Using the table again, we obtain $2 \oplus 3 = 5$.

Thus, according to the above two-step scheme, $4 \oplus 5 \oplus 3 = 5$. The two steps involved in obtaining this result can be indicated by using parentheses; that is, we can write

$$(4 \oplus 5) \oplus 3 = 5.$$

Parentheses have been put around $4 \oplus 5$ to show that 4 and 5 are grouped together and that their sum is to be added to 3.

What result would be obtained if we grouped in the following way?

$$4 \oplus (5 \oplus 3)$$

This time the parentheses indicate that 5 and 3 are to be added first and that the result is to be added to 4.

$$5 \oplus 3 = 1.$$
$$4 \oplus 1 = 5.$$

Combining the two steps, we write $4 \oplus (5 \oplus 3) = 5$. Notice that this is the same result we obtained before.

Class Discussion 5

1. Find the result in each of the following:

   a. $(6 \oplus 4) \oplus 5$
   b. $6 \oplus (4 \oplus 5)$

2. Notice that the order of the numbers in exercise 1a is the same as the order of the numbers in exercise 1b. That is, 6 is on the left, 4 is in the middle, and 5 is on the right. However, we are not particularly concerned here with the order of the numbers. In this instance we are concerned with the way in which the numbers are grouped. If a, b, and c represent any numbers in the set $\{0, 1, 2, 3, 4, 5, 6\}$, do you think that it is always true that

$$(a \oplus b) \oplus c = a \oplus (b \oplus c)?$$

3. If a, b, and c represent any numbers in the set $\{0, 1, 2, 3, 4, 5, 6\}$ and if it is always true that $(a \oplus b) \oplus c = a \oplus (b \oplus c)$, then we say that circle addition has the *associative property*. This means that when three given numbers are added, the final result will always be the same no matter how the numbers are grouped. For each pair of examples listed below, check whether or not the final results are the same.

   a. $(5 \oplus 6) \oplus 2$ and $5 \oplus (6 \oplus 2)$
   b. $3 \oplus (6 \oplus 6)$ and $(3 \oplus 6) \oplus 6$

4. Do you think that circle addition has the associative property?

5. What meaning may we attach to the expression $4 \oplus 5 \oplus 3$, where no grouping is indicated?

Exercises—5

1. Find the product $(3 \otimes 4) \otimes 5$. The parentheses are used as grouping symbols to indicate that 3 and 4 are to be multiplied first.

2. Find the product $3 \otimes (4 \otimes 5)$. Is the result the same as in exercise 1?

3. Is it true that $(2 \otimes 5) \otimes 4 = 2 \otimes (5 \otimes 4)$?

4. Is it true that $(5 \otimes 6) \otimes 3 = 5 \otimes (6 \otimes 3)$?

5. Is it true that $(2 \otimes 4) \otimes 6 = 2 \otimes (4 \otimes 6)$?

6. Do you think that circle multiplication has the associative property?

7. The results of the explorations in exercises 1-3 of Class Discussion 5 indicate clearly that circle addition is associative. So we conclude that in adding three numbers it does not matter how the numbers are grouped; the result will always be the same. But how shall we group to find the sum of four numbers, say $4 \oplus 5 \oplus 6 \oplus 3$? Here is one possibility: Begin by grouping the four numbers in the following way—$(4 \oplus 5 \oplus 6) \oplus 3$. Then group the three numbers in parentheses in one of two ways—either as $(4 \oplus 5) \oplus 6$ or as $4 \oplus (5 + 6)$. What result do you obtain if you use the scheme of grouping described?

8. Suppose you begin by grouping the four numbers in the following way—$4 \oplus (5 \oplus 6 \oplus 3)$—and then group the numbers within parentheses either as $(5 \oplus 6) \oplus 3$ or as $5 \oplus (6 \oplus 3)$. What result do you obtain if you use this scheme?

9. Suppose you begin by grouping the four numbers in the following way: $(4 \oplus 5) \oplus (6 \oplus 3)$. What result do you obtain this time?

10. On the basis of your experiences with exercises 7-9, what conclusion can you state concerning the results of different grouping schemes in the circle addition of four numbers?

11. In finding the sum $4 \oplus 5 \oplus 6 \oplus 3$ we could have used the number circle, moving first 4, then 5, then 6, and finally 3 spaces in the direction of the arrows. Try doing this. Do you obtain the same result as in exercises 7-9?

12. Find the sum $3 \oplus 4 \oplus 5 \oplus 6$. Then find the sum $4 \oplus 3 \oplus 6 \oplus 5$. Are the two results the same? Can the commutative property of circle addition be applied if more than two numbers are added?

13. Can the commutative property of circle multiplication be applied if more than two numbers are multiplied? Check your answer by finding the product in each of the following:

a. 3 ⓧ 4 ⓧ 5 ⓧ 6 **b.** 4 ⓧ 3 ⓧ 6 ⓧ 5

14. Arrange the four numbers in exercise 13a or 13b in any way that you please. Then find the product. Does the result that you obtain agree with the results that you obtained in exercise 13? Do you think these four numbers can be arranged in some way that will give you a different product?

15. Do ordinary addition and multiplication of whole numbers have both the commutative and associative properties? Make use of examples to justify your answer.

16. Undoubtedly you have occasionally been concerned with the problem of finding averages. Averages are used in everyday life in many ways. A baseball player has a batting average. For any two students there is always an average height. To find the average of two numbers, we first find the sum of the two numbers and then divide by 2. Find the average of 8 and 6; 20 and 32; 48 and 12.

17. We can indicate the operation of finding the average of two numbers by using the symbol ⓐ. Thus, 10 ⓐ 6 = 8. Find 25 ⓐ 35; 13 ⓐ 17; 24 ⓐ 34.

18. The operation ⓐ as defined in exercise 16 is a binary operation. Do you recall what binary means? (See the summary at the end of Section 2.) Although averages can be found for more than two numbers, we are using the symbol ⓐ to apply to only two numbers. Now suppose that we wish to determine what number, if any, is represented by

10 ⓐ 6 ⓐ 14.

Since ⓐ is a binary operation, we should consider grouping as we did in circle addition and circle multiplication. One way of grouping would be

(10 ⓐ 6) ⓐ 14.

By definition, 10 ⓐ 6 = 8, and 8 ⓐ 14 = 11. But suppose we group 10 ⓐ 6 ⓐ 14 in the following way: 10 ⓐ (6 ⓐ 14).

When the numbers are grouped in this way, we must first average 6 and 14 and then average the result with 10. But if we do this, the final result is 10.

a. Can you tell what number to assign to 10 ⓐ 6 ⓐ 14 if parentheses are missing?

b. Does the binary operation ⓐ have the associative property?

19. Perform the indicated operations in each of the following, using the grouping indicated by the parentheses.

a. $(8 - 2) - 1$

b. $8 - (2 - 1)$

20. Would you say that subtraction in ordinary arithmetic is associative?

21. Perform the operations indicated in each of the following, using the grouping indicated by the parentheses.

a. $(24 \div 6) \div 2$

b. $24 \div (6 \div 2)$

22. Does division in ordinary arithmetic have the associative property?

Summary—5

We have been exploring the system of circle arithmetic that involves the set of numbers $\{0, 1, 2, 3, 4, 5, 6\}$ and the operations of circle addition and circle multiplication.

Following exercise 4 in Class Discussion 3 we indicated that the set of numbers $\{0, 1, 2, 3, 4, 5, 6\}$ is closed with respect to circle addition and circle multiplication. In the present section our investigations led us to conclude that both circle addition and circle multiplication have the commutative and associative properties. Although we have considered the closure, commutative, and associative properties in this unit mainly in connection with circle addition and circle multiplication in the system of circle arithmetic, these properties apply as well to ordinary addition and multiplication of whole numbers.

To illustrate that operations do not always have the commutative and associative properties, two new operations were introduced. In this connection you should have discovered that the operation $\boxdot$, defined by the relation $a \boxdot b = 2a + b$, is not commutative, and that the averaging operation ⓐ is not associative.

In the next section you will see what happens if the operations of circle addition and circle multiplication are combined in a certain way.

6 A Matter of Distribution

What number is represented by the expression

$$3 \otimes (4 \oplus 6)?$$

The parentheses suggest that we might begin by finding the sum $4 \oplus 6$. The result of the circle addition in this case is 3. Since $3 \otimes 3 = 2$, the final result is 2.

But suppose that we multiply first and then add:

$$(3 \otimes 4) \oplus (3 \otimes 6)$$

As before, we do the work within parentheses first. Note that this time there are two sets of parentheses. We get

$$(3 \otimes 4) = 5 \quad \text{and} \quad (3 \otimes 6) = 4.$$

Adding the two results, we see that $5 \oplus 4 = 2$. Can we say that

$$3 \otimes (4 \oplus 6) = (3 \otimes 4) \oplus (3 \otimes 6)?$$

Class Discussion 6

1. Determine what number is represented by the expression $2 \otimes (5 \oplus 6)$. Begin by finding the sum within parentheses.
2. Determine what number is represented by the expression $(2 \otimes 5) \oplus (2 \otimes 6)$. Begin by finding the products in the two sets of parentheses.
3. Are the results you obtained in exercises 1 and 2 the same?

4. Can you say that $2 \otimes (5 \oplus 6) = (2 \otimes 5) \oplus (2 \otimes 6)$?
5. Show that $4 \otimes (5 \oplus 3) = (4 \otimes 5) \oplus (4 \otimes 3)$.
6. On the basis of the results obtained in the exercises above, write another expression that indicates the same number as $5 \otimes (2 \oplus 4)$.
7. Show that the expression you wrote in exercise 6 represents the same number as the given expression.
8. The results obtained in the exercises above indicate that there are two ways of finding the number represented by an expression such as $4 \otimes (2 \oplus 3)$. Describe the two ways.

The property that we have been investigating is referred to as the distributive property. In the system of circle arithmetic we can describe this property by saying *circle multiplication distributes over circle addition*. Formally we can state the distributive property for circle arithmetic in the following way:

If a, b, and c represent numbers in the set $\{0, 1, 2, 3, 4, 5, 6\}$, then it is always true that

$$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c).$$

Exercises—6

1. Perform the indicated operations in the expression $$4 \otimes (6 \oplus 3).$$
2. Show that $6 \otimes (5 \oplus 6) = (6 \otimes 5) \oplus (6 \otimes 6)$.
3. Show that $5 \otimes (6 \oplus 4) = (5 \otimes 6) \oplus (5 \otimes 4)$.
4. According to the distributive property, circle multiplication distributes over circle addition. Do you think circle addition distributes over circle multiplication?
5. Perform the indicated operations in $3 \oplus (4 \otimes 5)$. Do the computation within parentheses first.
6. Now perform the indicated operations in $(3 \oplus 4) \otimes (3 \oplus 5)$. Again, do the computation within parentheses first.
7. Are the results you obtained in exercises 5 and 6 the same?

8. Is it true that $3 \oplus (4 \otimes 5) = (3 \oplus 4) \otimes (3 \oplus 5)$?

9. Does circle addition distribute over circle multiplication?

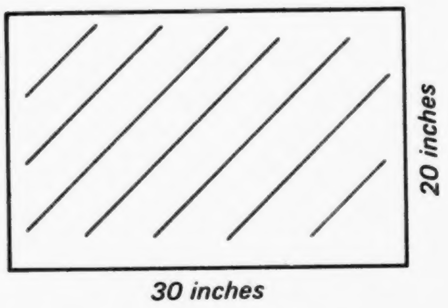
10. Consider now the set of whole numbers and ordinary multiplication and addition. Calculate $8 \times (10 + 5)$.

11. Calculate $(8 \times 10) + (8 \times 5)$. Is the result the same as in exercise 10?

12. Show that $12 \times (10 + 9) = (12 \times 10) + (12 \times 9)$. Does this example suggest that the distributive property applies to ordinary multiplication and addition of whole numbers?

7 Using the Distributive Property

In this section we shall explore ways of using the distributive property in computing with whole numbers. Pictured at the right is a mirror that has the shape of a rectangle. The mirror is 30 inches long and 20 inches wide. We want to know how many inches of framing material are needed to make a frame for the mirror.



One way to find out would be to make the following computations:

$$2 \times 30 = 60.$$
$$2 \times 20 = 40.$$
$$60 + 40 = 100.$$

Another way would be to find the sum of the length and the width of the mirror and then multiply the result by 2. If this plan is used, the required computation can be completed in the following way:

$$30 + 20 = 50.$$
$$2 \times 50 = 100.$$

Class Discussion 7

1. Do the two methods of finding the number of inches of framing material that are needed give us the same result?

2. Does $2 \times (30 + 20) = (2 \times 30) + (2 \times 20)$?

3. Does exercise 2 illustrate the use of the distributive property?

4. Use the distributive property to express each of the following as a sum of products:

a. $10 \times (8 + 9)$ **b.** $50 \times (12 + 8)$

5. Following is an example of the distributive property:

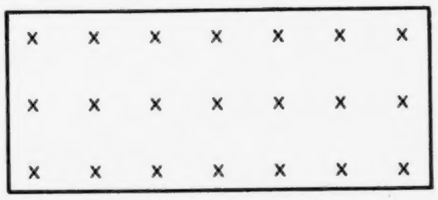
$$30 \times (5 + 7) = (30 \times 5) + (30 \times 7).$$

Study the form of this example carefully and then complete each of the following:

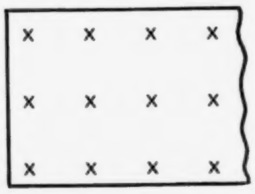
a. ____________ $= (25 \times 7) + (25 \times 12).$

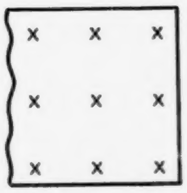
b. ____________ $= (35 \times 9) + (35 \times 2).$

6. In the chart below, explain how to determine the number of x's without counting all of them.



7. Suppose that the chart in exercise 6 is separated into two parts as shown below.





a. What shortcut can be used to find the number of x's in the part on the left?

b. What shortcut can be used to find the number of x's in the part on the right?

8. Why is the number of x's in the two parts pictured in exercise 7 the same as the number of x's in the chart pictured in exercise 6? Can you say that (3 x 4) + (3 x 3) = 3 x (4 + 3) = 3 x 7?

9. The distributive property is useful in doing multiplication mentally. For example, 8 x 62 can be thought of as 8 x (60 + 2). Then, using the distributive property, we see that 8 x (60 + 2) is equal to (8 x 60) + (8 x 2), which is 480 + 16 = 496. Do the multiplication in each example below mentally. Use the procedure explained above if you think it will simplify the task.

a. 5 x 41 **b.** 6 x 32 **c.** 4 x 63 **d.** 7 x 42

Exercises—7

The examples of the distributive property for ordinary multiplication and addition of whole numbers which have been presented thus far can be expressed somewhat more conveniently if we indicate multiplication in other ways than by using the symbol "x." For example, we can express 30 x 5 by writing either 30 · 5 or 30(5). If we use the last two forms for indicating multiplication, the example

$$30 \times (5 + 7) = (30 \times 5) + (30 \times 7)$$

can be rewritten in the form

$$30\,(5 + 7) = (30 \cdot 5) + (30 \cdot 7).$$

In doing the exercises in this section, express multiplication in any way you prefer.

1. Find the answer for each exercise listed below. Apply the distributive property whenever you think it will make the work easier.

a. 5 (4 + 7)
b. 16 (22 + 12)
c. 12 (14 + 8)
d. 8 (25 + 37)
e. 7 (23 + 10)
f. 15 (34 + 46)

2. To find the product 8 · 56 mentally, recall that the task can be made easier if you first think of 8 · 56 as being equal to 8 (50 + 6). (See exercise 9 in Class Discussion 7.) Make use of this scheme in finding the following products:

a. $6 \cdot 45$
b. $3 \cdot 39$
c. $4 \cdot 52$
d. $3 \cdot 71$
e. $7 \cdot 43$
f. $8 \cdot 32$

3. Find each of the products below. Using the distributive property will usually make the task easier.

a. $2 \cdot 46$
b. $5 \cdot 43$
c. $7 \cdot 35$
d. $9 \cdot 27$
e. $6 \cdot 34$
f. $5 \cdot 67$

Summary—7

In exploring the system of circle arithmetic in this unit, we have investigated rather thoroughly certain properties of circle addition and circle multiplication. Summarized below are statements of the properties of circle addition and circle multiplication which have been studied thus far. Let a, b, and c represent numbers in the set {0, 1, 2, 3, 4, 5, 6}.

Operations ⟶	Circle Addition	Circle Multiplication
Closure properties ⟶	$a \oplus b$ is a number in the set {0, 1, 2, 3, 4, 5, 6}.	$a \otimes b$ is a number in the set {0, 1, 2, 3, 4, 5, 6}.
Commutative properties ⟶	$a \oplus b = b \oplus a$.	$a \otimes b = b \otimes a$.
Associative properties ⟶	$(a \oplus b) \oplus c = a \oplus (b \oplus c)$.	$(a \otimes b) \otimes c = a \otimes (b \otimes c)$.
Distributive property ⟶	$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$.	

In working with circle arithmetic we have noted that properties like those listed above also apply to ordinary addition and multiplication of whole numbers. To make things definite, let us restate the above properties in language that is appropriate to operations with whole numbers. In the statements that follow, a, b, and c represent arbitrary whole numbers. That is, a, b, and c represent numbers in the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ... }.

1. *The closure property of addition.* The sum of any two whole numbers is always a whole number. (For any whole number a and any whole number b, there is a whole number c such that $a + b = c$.)

2. *The closure property of multiplication.* The product of any two whole numbers is always a whole number. (For any whole number a and any whole number b, there is a whole number c such that $a \cdot b = c$.)
3. *The commutative property of addition.* The order in which any two whole numbers are added does not affect the sum. (For any whole number a and any whole number b, it is true that $a + b = b + a$.)
4. *The commutative property of multiplication.* The order in which any two whole numbers are multiplied does not affect the product. (For any whole number a and any whole number b, it is true that $a \cdot b = b \cdot a$.)
5. *The associative property of addition.* The way in which three whole numbers are grouped does not affect the sum. [For any whole number a, any whole number b, and any whole number c, it is true that $(a + b) + c = a + (b + c)$.]
6. *The associative property of multiplication.* The way in which three whole numbers are grouped does not affect the product. [For any whole number a, any whole number b, and any whole number c, it is true that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.]
7. *The distributive property of multiplication over addition.* Multiplication of whole numbers is distributive with respect to addition of whole numbers. If the sum of two whole numbers is to be multiplied by a third whole number, then instead of first adding and then multiplying, we can first multiply each of the first two numbers by the third number and then add. [For any whole number a, any whole number b, and any whole number c, it is true that $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.]

8 A Question of Identity

Thus far in this unit you have studied certain properties of circle addition and circle multiplication, and also the corresponding properties of ordinary addition and multiplication of whole numbers.

The properties that you studied are reviewed in the summary above.

In the present section you will have an opportunity to investigate some properties of addition and multiplication that concern the numbers 0 and 1.

Up until now you first investigated a particular property in the system of circle arithmetic and then looked for a corresponding property in the ordinary arithmetic of whole numbers. In this and the remaining sections of this unit, the procedure is turned around. From now on you will first investigate a particular property in ordinary arithmetic and then look for a corresponding property in circle arithmetic.

Class Discussion **8**

1. Suppose that when you add two whole numbers, the sum that you get equals one of the two numbers that you added. Name the other number.

2. Does $7 + 0 = 7$? Does $100 + 0 = 100$? Does $0 + 0 = 0$? Is the sum of any given whole number and 0 the same number as the given whole number?

3. Your answers to the questions in exercise 2 should convince you that *for any whole number a, it is true that* $a + 0 = a$. The underlined statement is referred to as the *addition property of zero.* Could this property also be stated in the following way: "For any whole number a, it is true that $0 + a = a$"? Explain your answer.

4. Because the sum of any whole number a and 0 is identical with the whole number a, we refer to the number 0 as an *additive identity*. Do you think there is more than one additive identity in the set of whole numbers? Or do you think that 0 is the only additive identity in the set of whole numbers?

5. Suppose that you multiply 7 by a certain whole number and get 7 for your answer. What number did you multiply by?

6. What is the product of 25 and 1? What is the product of 40 and 1? Is the product of any given whole number and 1 equal to the given whole number?

7. Because the product of 1 and any given whole number is identically equal to the given whole number, we refer to the number 1 as a *multiplicative identity*. Do you think that there is more than one multiplicative identity in the set of whole numbers? Or do you think that 1 is the only multiplicative identity in the set of whole numbers?

8. The discussion in exercises 5-7 indicates clearly that multiplication by 1 has a certain property. Let us call this property the *multiplication property of 1*. Try stating this property. (*Hint:* See exercise 3 above.)

9. The addition property of zero tells us that for any whole number a, it is true that $a + 0 = a$. Do you think that addition of any whole number and 1 has some special property?

10. The multiplication property of 1 tells us that for any whole number a, it is true that $a \cdot 1 = a$. Do you think that multiplication of any whole number and 0 has some special property? What is the product of any whole number and 0? Does your answer to the last question suggest a property? Try stating this property. Would *multiplication property of zero* be an appropriate name for this property?

11. Now let us ask the question, Is there a whole number a such that

$$5 + a = 0?$$

Can you find a whole number b such that

$$4 + b = 0?$$

12. Can you find a whole number c such that

$$5 \cdot c = 1?$$

Is there a whole number d such that

$$4 \cdot d = 1?$$

13. Do you see that in ordinary arithmetic there are no whole numbers a, b, c, or d that make any of the following sentences true?

$$5 + a = 0. \qquad 4 + b = 0. \qquad 5 \cdot c = 1. \qquad 4 \cdot d = 1.$$

14. Do you think the same situation exists in circle arithmetic? This question is investigated in the exercises that follow.

Exercises—8

1. Using your table for circle addition or the number circle, complete each of the following so that the resulting sentence is true:

a. $6 \oplus \underline{\qquad} = 0.$
b. $5 \oplus \underline{\qquad} = 0.$
c. $4 \oplus \underline{\qquad} = 0.$
d. $3 \oplus \underline{\qquad} = 0.$
e. $2 \oplus \underline{\qquad} = 0.$
f. $1 \oplus \underline{\qquad} = 0.$
g. $0 \oplus \underline{\qquad} = 0.$

2. Using your table for circle multiplication, complete each of the following so that the resulting sentence is true:

a. $6 \otimes \underline{\qquad} = 1.$
b. $5 \otimes \underline{\qquad} = 1.$
c. $4 \otimes \underline{\qquad} = 1.$
d. $3 \otimes \underline{\qquad} = 1.$
e. $2 \otimes \underline{\qquad} = 1.$
f. $1 \otimes \underline{\qquad} = 1.$

3. Each of exercises 1a through 1g involves circle addition of two numbers in the set $\{0, 1, 2, 3, 4, 5, 6\}$; the sum in each exercise is 0. If two numbers have the sum 0, each is the *additive inverse* of the other. For example, $6 \oplus 1 = 0$. So 6 and 1 are additive inverses of each other. In the system of circle arithmetic, what is the additive inverse of 2; 3; 4; 5?

4. What is the additive inverse of 0?

5. In the system of circle arithmetic, does every number in the set $\{0, 1, 2, 3, 4, 5, 6\}$ have an additive inverse?

6. Each of exercises 2a through 2f involves circle multiplication of two numbers in the set $\{0, 1, 2, 3, 4, 5, 6\}$; the product in each case is 1. If two numbers have the product 1, each is the *multiplicative inverse* of the other. Since $3 \otimes 5 = 1$, 3 and 5 are multiplicative inverses of each other. In the system of circle arithmetic, what is the multiplicative inverse of 1; 2; 4; 6?

7. Does 0 have a multiplicative inverse in the system of circle arithmetic?

8. In the system of circle arithmetic does every number except 0 have a multiplicative inverse?

9. In ordinary addition of whole numbers, does every whole number have an additive inverse that is also a whole number? Is there any whole number that has an additive inverse in the set of whole numbers?

10. In ordinary multiplication of whole numbers, does every whole number, except 0, have a multiplicative inverse that is also a whole number? Is there any whole number that has a multiplicative inverse in the set of whole numbers?

9 Subtraction on the Circle

Earlier in this unit we noted that in ordinary arithmetic the set of whole numbers is closed with respect to addition and multiplication. Is the set of whole numbers closed with respect to subtraction?

You already know how to subtract whole numbers. So you know, for example, that $8 - 5 = 3$ and that $10 - 4 = 6$. On the other hand, $4 - 7$ is meaningless in the ordinary arithmetic of whole numbers.

One example in which subtraction is meaningless is sufficient to show that the set of whole numbers is not closed with respect to subtraction. But why do we say that $4 - 7$ is meaningless? To answer this question, we are forced to think about the meaning of subtraction. The usual check on subtraction by addition gives us a clue as to what subtraction means. In checking subtraction we usually reason like this:

$$8 - 5 = 3 \quad \text{because} \quad 3 + 5 = 8.$$

In other words, $8 - 5$ can be thought of as a number such that the sum of this number and 5 is 8.

What does $10 - 4$ mean? According to the interpretation given above, $10 - 4$ is a number such that the sum of this number and 4 is 10. The last statement can be made clearer by saying $10 - 4$ is a number b such that $b + 4 = 10$. (The number b is called the *difference* of 10 and 4.) Thus we can say $10 - 4 = b$ means that $b + 4 = 10$.

Let us look again at the example $4 - 7$. Is there a whole number b such that $b + 7 = 4$? We know, of course, there is not. Therefore we know there is no whole number b such that $4 - 7 = b$. It is because there is no whole number b such that $4 - 7 = b$ that we say $4 - 7$ is meaningless in the ordinary arithmetic of whole numbers.

Let us consider one other example. Is there a whole number b such that $6 - 8 = b$? If there is, then the whole number b must make the sentence $b + 8 = 6$ true. But there is no whole number to which we can add 8 and get 6. Hence, the difference of 6 and 8 is not a whole number.

Class Discussion 9

1. Now let us investigate whether or not the set of numbers $\{0, 1, 2, 3, 4, 5, 6\}$ in the system of circle arithmetic is closed with respect to circle subtraction. We indicate circle subtraction as you would expect—by a "minus" symbol with a circle around it $\ominus$. What number is represented by $6 \ominus 4$? Reasoning as we did above, we ask the question, Is there a number b in the set $\{0, 1, 2, 3, 4, 5, 6\}$ such that $b \oplus 4 = 6$? Does $6 \ominus 4 = 2$?

2. Does $2 \ominus 5$ represent a number in the set $\{0, 1, 2, 3, 4, 5, 6\}$ of the system of circle arithmetic? If it does, then there is some number b in the set such that $b \oplus 5 = 2$. What number is represented by b? Make use of your table for circle addition to find the answer.

3. What number is represented by $4 \ominus 6$?

4. Complete the sentences below.

a. $2 \ominus 6 =$ ____ because ____ $\oplus\ 6 = 2$.
b. $3 \ominus 4 =$ ____ because ____ $\oplus\ 4 = 3$.
c. $2 \ominus 3 =$ ____ because ____ $\oplus\ 3 =$ ____.
d. ____ $\ominus\ 6 = 1$ because ____ $\oplus$ ____ $= 0$.
e. $4 \ominus$ ____ $= 5$ because ____ $\oplus$ ____ $= 4$.

5. Do you think that the set of numbers $\{0, 1, 2, 3, 4, 5, 6\}$ is closed with respect to circle subtraction?

Exercises—9

1. In the system of circle arithmetic what number is represented by each of the following?

a. $3 \ominus 5$
b. $2 \ominus 4$
c. $0 \ominus 5$
d. $3 \ominus 6$
e. $1 \ominus 4$
f. $5 \ominus 4$
g. $0 \ominus 3$
h. $0 \ominus 4$
i. $1 \ominus 5$
j. $5 \ominus 6$
k. $0 \ominus 6$
l. $2 \ominus 6$

2. In circle arithmetic what number must be added to 6 to obtain 3?

3. What number must be added to 3 to obtain 0?

4. Since $3 \oplus 4 = 0$, we know that in the system of circle arithmetic 4 is the additive inverse of 3 and 3 is the additive inverse of 4. (See Exercises—8.) In general, if a represents any number in the set $\{0, 1, 2, 3, 4, 5, 6\}$ and b represents any number in this set such that $a \oplus b = 0$, then the numbers represented by a and b are additive inverses of each other. In the system of circle arithmetic, what is the additive inverse of each number in the set $\{0, 1, 2, 3, 4, 5, 6\}$?

5. If you obtained the correct answer in exercise 1b above, you found that $2 \ominus 4 = 5$. What is the additive inverse of 4? What is the sum $2 \oplus 3$?

6. Note that $2 \ominus 4 = 2 \oplus 3$. Since 3 is the additive inverse of 4, what scheme does this example suggest as a way of doing subtraction in circle arithmetic?

7. From exercise 1d you know that $3 \ominus 6 = 4$. You also know that $3 \oplus 1 = 4$. By looking at these two examples you can see that 1 and 6 are additive inverses of each other. Change each of exercises 1a through 1k to a circle-addition exercise having the same answer as the given exercise. Then do the addition. Example: $3 \ominus 5 = 3 \oplus 2 = 5$.

10 Division on the Circle

Division and multiplication are related in somewhat the same way as subtraction and addition. For example, we say that

$$21 \div 3 = 7 \quad \text{because} \quad 7 \cdot 3 = 21.$$

In the division example on the left, 21 is referred to as the *dividend*, 3 as the *divisor*, and 7 as the *quotient*. To divide a number (the dividend) by a number (the divisor) means to find a number (the quotient) such that

$$\text{quotient} \times \text{divisor} = \text{dividend}.$$

There is one number that cannot be used as a divisor. This number is zero. In the exercises below you will learn why division by zero is ruled out.

Here is another way of stating what division means: If a, b, and c represent numbers, and b is not zero, then

$$a \div b = c \quad \text{means} \quad c \cdot b = a.$$

For example, to divide 42 by 7 you need to find a number c such that $c \cdot 7 = 42$.

Class Discussion 10

1. You know that $9 \times 3 = 27$. How does this fact help you find the quotient of 27 and 3?
2. Is there a whole number c such that $36 \div 9 = c$? If there is, then the product of this number and 9 must equal 36.
3. For what whole number c is it true that $c \cdot 7 = 63$? Does this number equal $63 \div 7$?
4. For what whole number c is it true that $c \cdot 6 = 17$? Does $17 \div 6$ give you a whole number for an answer?
5. Is the set of whole numbers closed with respect to division in ordinary arithmetic?
6. Show by using multiplication that $0 \div 6 = 0$.
7. Is there a whole number that equals $0 \div 100$? How do you know?

8. Do you think that $0 \div b = 0$ for every whole number b that is not zero?

9. **a.** Is there a whole number c for which it is true that $6 \div 0 = c$? If there is, then it must also be true that $c \cdot 0 = 6$. Why? Is the product of any whole number and 0 equal to 6? If b is not zero, is there a whole number c for which it is true that $b \div 0 = c$? Explain your answer.

b. What number does $0 \div 0$ represent? According to the meaning we have given to division, $0 \div 0$ is a number c, if there is one, which makes it true that $c \cdot 0 = 0$. For how many whole numbers c is it true that $c \cdot 0 = 0$? (*Hint:* From exercise 10, page 27, you know that the product of every whole number and zero equals zero.) Do you see that $0 \div 0$ can be any number? Thus, the quotient $0 \div 0$ is not useful.

Your answers to the questions in exercises 9a and 9b should help you see why division by zero is not allowed.

10. Division in the system of circle arithmetic has the same meaning as division in ordinary arithmetic. If a, b, and c represent numbers in the set $\{0, 1, 2, 3, 4, 5, 6\}$, and b is not zero, then

$$a \odiv b = c \quad \text{means that} \quad c \otimes b = a.$$

You know that $3 \otimes 2 = 6$. Does this tell you that $6 \odiv 2 = 3$?

11. Is $5 \odiv 4$ equal to a number in the set $\{0, 1, 2, 3, 4, 5, 6\}$? In other words, is there a number c in this set such that $5 \odiv 4 = c$? If there is, then $c \otimes 4 = 5$.

12. What number is represented by $4 \odiv 3$? To answer this question, you need to find a number c such that $c \otimes 3 = 4$.

13. What number is represented by $2 \odiv 5$? Is this number in the set $\{0, 1, 2, 3, 4, 5, 6\}$? What number is represented by $6 \odiv 0$?

14. From exercise 10 you know that division by zero is not allowed in circle arithmetic. The reason for ruling it out is the same as for ruling it out in the arithmetic of whole numbers. Suppose, then, that we leave 0 out of the set $\{0, 1, 2, 3, 4, 5, 6\}$. Do you think the set $\{1, 2, 3, 4, 5, 6\}$ is closed with respect to circle division?

Exercises—10

1. By checking your table for circle multiplication you can see that 3 ⊗ 4 = 5. Give an example involving circle division that is related to this circle-multiplication example. (See exercise 10 in Class Discussion 10.)

2. Complete each of the following so that the resulting sentence is true.

a. ____ ⊗ 2 = 5.
b. ____ ⊗ 2 = 3.
c. ____ ⊗ 4 = 1.
d. ____ ⊗ 4 = 6.
e. ____ ⊗ 3 = 2.
f. ____ ⊗ 5 = 4.
g. ____ ⊗ 6 = 3.
h. ____ ⊗ 5 = 3.
i. ____ ⊗ 5 = 1.
j. ____ ⊗ 3 = 1.

3. Change each completed sentence in exercise 2 to a related sentence involving circle division. For example, 6 ⊗ 2 = 5 in exercise 2a becomes 5 ⨸ 2 = 6.

4. The completed sentences in exercises 2i and 2j should be 3 ⊗ 5 = 1 and 5 ⊗ 3 = 1. Are 3 and 5 multiplicative inverses of each other in the system of circle arithmetic?

5. What pairs of numbers in the set $\{1, 2, 3, 4, 5, 6\}$ are multiplicative inverses of each other in the system of circle arithmetic? Is any number in this set its own multiplicative inverse?

6. From the exercises above you know that 4 ⨸ 3 = 6, that 4 ⊗ 5 = 6, and that 5 is the multiplicative inverse of 3. Do these facts suggest a special way of doing circle division?

7. To find the quotient in each exercise below, replace the divisor by its multiplicative inverse; then find the product of the two numbers. That is, find the product of the dividend and the multiplicative inverse of the divisor. Example: 3 ⨸ 4 = 3 ⊗ 2 = 6.

a. 5 ⨸ 2
b. 6 ⨸ 5
c. 4 ⨸ 5
d. 2 ⨸ 6
e. 1 ⨸ 4
f. 1 ⨸ 5
g. 5 ⨸ 3
h. 6 ⨸ 4
i. 1 ⨸ 2
j. 5 ⨸ 4

General Summary

In this unit we explored a mathematical system called circle arithmetic. As was explained at the end of Section 2, there are various such systems. The particular system that was introduced in this unit consists of the set of numbers $\{0, 1, 2, 3, 4, 5, 6\}$ and the two binary operations circle addition and circle multiplication. On the basis of the explorations that were made we concluded that the set of numbers $\{0, 1, 2, 3, 4, 5, 6\}$ is closed with respect to circle addition and circle multiplication, that both operations have the commutative and associative properties, and that circle multiplication distributes over circle addition.

Also investigated were the special properties of circle addition and circle multiplication that concern the numbers 0 and 1. By means of examples it was shown that if a represents any number in the set $\{0, 1, 2, 3, 4, 5, 6\}$, then

$$a \oplus 0 = a,$$
$$a \otimes 1 = a,$$

and

$$a \otimes 0 = 0.$$

Each property described above for the system of circle arithmetic was also investigated in connection with ordinary arithmetic involving the set of whole numbers $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \dots\}$. By means of examples it was shown that ordinary addition and multiplication of whole numbers have properties that are entirely analogous to those described above.

Further explorations revealed that there are certain properties in the system of circle arithmetic for which there are no corresponding properties in the ordinary arithmetic of whole numbers. The circle-arithmetic properties to which we have reference are listed below.

1. The set $\{0, 1, 2, 3, 4, 5, 6\}$ is closed with respect to circle subtraction.
2. Every number in the set $\{0, 1, 2, 3, 4, 5, 6\}$ has an additive inverse.
3. The set of numbers $\{1, 2, 3, 4, 5, 6\}$ is closed with respect to circle division.

4. Every number in the set $\{1, 2, 3, 4, 5, 6\}$ has a multiplicative inverse.

Review Exercises

1. Copy and complete each of the following tables for circle arithmetic.

a.

⊕	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

b.

⊗	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

2. What property of circle addition is illustrated by the example $3 \oplus 4 = 4 \oplus 3$?

3. What property of circle multiplication is illustrated by the example $2 \otimes 5 = 5 \otimes 2$?

4. Show by carrying out the indicated operations that the sentence $(3 + 4) + 6 = 3 + (4 + 6)$ is true.

5. What property is illustrated by exercise 4?

6. Does this same property apply to circle addition?

7. Show by carrying out the indicated operations that the sentence $6 \times (3 + 5) = (6 \times 3) + (6 \times 5)$ is true. What property is illustrated by the given sentence?

8. For each of the following, identify the property from ordinary arithmetic that is illustrated.

a. $8(3 + 4) = 24 + 32$.
b. $7 + 2 = 2 + 7$.
c. $5 \cdot 6 = 6 \cdot 5$.
d. $(4 + 3) + 2 = 4 + (3 + 2)$.
e. $14 \cdot 72 = 72 \cdot 14$.
f. $(18 \cdot 7) \cdot 6 = 18 \cdot (7 \cdot 6)$.

9. Complete each of the following so that the resulting sentence is true. In each case name the property that you apply in completing the sentence.

a. $9 \cdot 8 = 8 \cdot \_\_$.
b. $4 + 5 = \_\_ + 4$.
c. $5(4 + 9) = \_\_ + \_\_$.
d. $(4 + 5) + 8 = 4 + (\_\_ + \_\_)$.
e. $(20 \cdot 5) \cdot 6 = (\_\_ \cdot 20) \cdot \_\_$.

10. Complete each of the following so that the resulting sentence is true.

a. $2 \cdot (6 + 3) = (2 \cdot \_\_) + (2 \cdot \_\_)$.
b. $3 \cdot (7 + 1) = (3 \cdot \_\_) + (\_\_ \cdot 1)$.
c. $5 \cdot (2 + \_\_) = (\_\_ \cdot 2) + (\_\_ \cdot 4)$.
d. $4 \cdot (\_\_ + 3) = (\_\_ \cdot 8) + (4 \cdot \_\_)$.
e. $\_\_(4 + 9) = (\_\_ \cdot 4) + (2 \cdot \_\_)$.
f. $\_\_(3 + \_\_) = (8 \cdot \_\_) + (\_\_ \cdot 7)$.
g. $8(2 + 6) = (\_\_ \cdot \_\_) + (\_\_ \cdot \_\_)$.
h. $7(\_\_ + \_\_) = (\_\_ \cdot 3) + (\_\_ \cdot 4)$.
i. $(6 \cdot 3) + (6 \cdot 9) = \_\_(\_\_ + \_\_)$.
j. $(\_\_ \cdot 2) + (9 \cdot \_\_) = \_\_(\_\_ + 8)$.

11. Show that the sentences below are true by carrying out the indicated operations.

a. $3 \cdot (2 + 4) = (3 \cdot 2) + (3 \cdot 4)$.
b. $6 \cdot (7 + 1) = (6 \cdot 7) + (6 \cdot 1)$.
c. $4 \cdot (3 + 5) = (3 \cdot 4) + (5 \cdot 4)$.
d. $(5 \cdot 6) + (5 \cdot 4) = 5(6 + 4)$.

12. For each exercise, find the result of the indicated circle operation.

a. $3 \ominus 5$
b. $4 \ominus 6$
c. $2 \odiv 3$
d. $1 \odiv 5$
e. $1 \ominus 5$
f. $4 \ominus 5$
g. $3 \odiv 4$
h. $5 \odiv 6$
i. $6 \odiv 5$
j. $2 \ominus 5$

13. In the system of circle arithmetic, what number in the set {0, 1, 2, 3, 4, 5, 6} does not have a multiplicative inverse?

14. In the system of circle arithmetic, what is the additive inverse of each number in the set {0, 1, 2, 3, 4, 5, 6}?

15. In the system of circle arithmetic, what is the multiplicative inverse of each number in the set {1, 2, 3, 4, 5, 6}?

16. In the ordinary arithmetic of whole numbers, what whole number has an additive inverse that is a whole number? What whole number has a multiplicative inverse that is a whole number?

3

Mathematical Sentences

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UNIT THREE OF

Experiences in Mathematical Discovery

Mathematical Sentences

NCTM
NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

1201 Sixteenth Street, N.W., Washington, D.C. 20036

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Preface

"Experiences in Mathematical Discovery" is a series of ten self-contained units, each of which is designed for use by students of ninth-grade general mathematics. These booklets are the culmination of work undertaken as part of the General Mathematics Writing Project of the National Council of Teachers of Mathematics (NCTM).

The titles in the series are as follows:

Unit 1: *Formulas, Graphs, and Patterns*
Unit 2: *Properties of Operations with Numbers*
Unit 3: *Mathematical Sentences*
Unit 4: *Geometry*
Unit 5: *Arrangements and Selections*
Unit 6: *Mathematical Thinking*
Unit 7: *Rational Numbers*
Unit 8: *Ratios, Proportions, and Percent*
Unit 9: *Measurement*
Unit 10: *Positive and Negative Numbers*

This project is experimental. Teachers may use as many units as suit their purposes. Authors are encouraged to develop similar approaches to the topics treated here, and to other topics, since the aim of the NCTM in making these units available is to stimu-

late the development of special materials that can be effectively used with students of general mathematics.

Preliminary versions of the units were produced by a writing team that met at the University of Oregon during the summer of 1963. The units were subsequently tried out in ninth-grade general mathematics classes throughout the United States.

Oscar F. Schaaf, of the University of Oregon, was director of the 1963 summer writing team that produced the preliminary materials. The work of planning the content of the various units was undertaken by Thomas J. Hill, Oklahoma City Public Schools, Oklahoma City, Oklahoma; Paul S. Jorgensen, Carleton College, Northfield, Minnesota; Kenneth P. Kidd, University of Florida, Gainesville, Florida; and Max Peters, George W. Wingate High School, Brooklyn, New York.

The Advisory Committee for the General Mathematics Writing Project was composed of Emil J. Berger (*chairman*), St. Paul Public Schools, St. Paul, Minnesota; Irving Adler, North Bennington, Vermont; Stanley J. Bezuszka, S.J., Boston College, Chestnut Hill, Massachusetts; Eugene P. Smith, Wayne State University, Detroit, Michigan; and Max A. Sobel, Montclair State College, Upper Montclair, New Jersey.

The following people participated in the writing of either the preliminary or the revised versions of the various units:

Donald H. Firl, Rochester Public Schools, Rochester, Minnesota
Carroll E. Fogal, University of Florida, Gainesville, Florida
Edward F. Gottlieb, Madison High School, Portland, Oregon
Kenneth P. Kidd, University of Florida, Gainesville, Florida
Edna K. Lassar, Polytechnic Institute of Brooklyn, Brooklyn, New York
John F. LeBlanc, Racine Public Schools, Racine, Wisconsin
Scott D. McFadden, Spencer Butte Junior High School, Eugene, Oregon
Max Peters, George W. Wingate High School, Brooklyn, New York

Leonard Simon, Bureau of Curriculum Research, New York Public Schools, New York, New York

Irwin N. Sokol, Collinwood High School, Cleveland, Ohio

William H. Spragens, University of Louisville, Louisville, Kentucky

Mary E. Stine, Fairfax County Schools, Fairfax, Virginia

Arthur J. Wiebe, Pacific College, Fresno, California

John E. Yarnelle, Hanover College, Hanover, Indiana

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EMIL J. BERGER
Chairman, Advisory Committee
General Mathematics Writing Project

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Experiences in Mathematical Discovery

3

Mathematical Sentences

1 Mathematical Sentences

$3 + 5 = 8.$

$2 + 4 = 9.$

$x + 2 = 5.$

The three mathematical sentences shown at the right were written on the chalkboard in Mr. Smith's classroom. Some of the students looked at the second sentence and smiled.

"Can it be," they wondered, "that at last Mr. Smith has made a mistake?"

Mr. Smith explained that the sentences on the chalkboard were examples of three basic types of mathematical sentences.

We know that the first sentence expresses a true idea so we call it a ***true sentence***. The second sentence, on the other hand, expresses a false idea, so we call it a *false sentence*. The third sentence is neither true nor false, since we do not know what number the variable x represents. Sentences of the third kind are called *open sentences*. We call them open sentences because the variable is open for replacement by a number. In this unit we shall use only whole numbers (0, 1, 2, 3, and so on) as replacements for a variable.

There are many kinds of open sentences. Some of these have special names. For example, an open sentence like the third one displayed above, which includes the symbol for "equals" (=), is generally called an equation. However, in this unit we shall consistently refer to a mathematical sentence of this kind as an open sentence.

If the variable x in the open sentence $x + 2 = 5$ is replaced by 3, then the resulting sentence is true. Accordingly we say that 3 is a *solution* of $x + 2 = 5$. Because 3 is the only solution of $x + 2 = 5$, we refer to 3 as *the* solution of $x + 2 = 5$.

Class Discussion 1

1. Classify each of the following sentences as true, false, or open:

a. $2 + 5 = 7$. **c.** $7 - 5 = 3$. **e.** $5 + 7 = 12$.
b. $2x + 4 = 8$. **d.** $3 - x = 2$. **f.** $10 - 4 = 6$.

2. Make a guess as to what *the* solution is in each of the following open sentences. (When we ask for *the* solution, you may assume that there is exactly one solution.)

a. $x + 8 = 11$. **c.** $2x + 4 = 10$. **e.** $w - 9 = 6$.
b. $x - 3 = 6$. **d.** $3y - 1 = 5$. **f.** $3x - 4 = 5$.

3. Check the guesses that you made above in exercises 2a through 2f by replacing the variable in each open sentence by the number that you guessed to be the solution. For example, if you guessed the solution in exercise 2a to be 3, replace x by 3 in the open sentence $x + 8 = 11$. Since $3 + 8 = 11$ is a true sentence, you know that the solution of $x + 8 = 11$ is 3.

4. Suppose that you replace the variable in each of the following open sentences by 2. Which of the resulting sentences will be true?

a. $x + 5 = 8$. **b.** $2y - 1 = 7$. **c.** $3x + 4 = 10$.

Summary—1

In this section you studied three kinds of mathematical sentences —true sentences, false sentences, and open sentences. An open sentence is a mathematical sentence that contains one or more variables. Each open sentence that we considered in this section contains exactly one variable. If the variable in such an open sentence is replaced by a number, the sentence that you get is either a true sentence or a false sentence.

A number that is used as a replacement for a variable in an open sentence and which makes the resulting sentence true is called a solution of the open sentence. If there is exactly one number that makes an open sentence true, we usually refer to this number as *the* solution of the open sentence.

Exercises—1

1. Give two examples of open sentences.

2. Classify the following sentences as true, false, or open.

a. $3 - 2 = 5$.
b. $7 + 4 = 11$.
c. $5 - x = 1$.
d. $2x - 3 = 7$.

3. Find the solution of each of the following open sentences by guessing and checking.

a. $x + 8 = 19$.
b. $2x + 7 = 17$.
c. $x - 8 = 13$.
d. $3x + 4 = 19$.
e. $2y - 3 = 11$.
f. $4x + 7 = 39$.
g. $3y - 5 = 16$.
h. $7x + 3 = 52$.
i. $5m - 6 = 29$.
j. $8x + 9 = 73$.

4. For each of the open sentences listed below, replace the variable by the number indicated and decide whether the resulting sentence is true or false.

a. $3x - 5 = 13$; replace x by 6.
b. $7y + 3 = 80$; replace y by 10.
c. $5w - 8 = 2$; replace w by 3.
d. $4g + 17 = 45$; replace g by 7.
e. $7m + 12 = 54$; replace m by 5.

5. Find the solution for each of the following open sentences.

a. $2x - 11 = 3$.
b. $5x + 9 = 34$.
c. $4x - 8 = 20$.
d. $9x - 7 = 20$.
e. $3y + 3 = 30$.
f. $5y - 5 = 25$.
g. $8m + 7 = 71$.
h. $5m + 8 = 73$.
i. $2z - 7 = 41$.
j. $12x + 7 = 79$.

6. In each open sentence listed below, the same variable occurs more than once. Find the solution of each sentence. (If the same variable occurs more than once in an open sentence, then it must be replaced by the same number each time it occurs.)

- **a.** $2x + 3 = x + 8$.
- **b.** $3y - 4 = y + 2$.
- **c.** $2m - 5 = m + 3$.
- **d.** $4x - 5 = 2x - 1$.
- **e.** $2y + 2 = y + 4$.

2 Doing and Undoing

Phil and Mary were asked to demonstrate the meaning of undoing an action. Mary was asked to undo what Phil had done. The following examples are illustrative.

Phil opened the window.	*Mary closed the window.*
Phil wrote on the board.	*Mary erased the writing.*
Phil dropped a book.	*Mary picked up the book and gave it to Phil.*

Class Discussion 2a

1. Describe Mary's action of undoing if Phil had—

- **a.** Closed the door.
- **b.** Moved a chair two feet forward.
- **c.** Picked up an eraser.

2. For each of the following, describe the action that is needed to undo the action that is indicated, provided that an undoing action is possible.

- **a.** Opening a book
- **b.** Walking two steps backward
- **c.** Taking off a shoe
- **d.** Tearing a picture
- **e.** Combing your hair
- **f.** Cutting your finger

3. An action may consist of several steps. In such instances, the undoing action may also involve several steps. Describe the steps that are needed to undo each of the following.
   - a. Taking a book from the shelf and opening it
   - b. Picking up a pencil and putting it in a pair of compasses
   - c. Opening a notebook and removing a sheet of paper
   - d. Standing up, walking to the board, and picking up a piece of chalk
4. The notion of undoing can also be applied to operations with numbers. Describe the undoing operation for each of the following.
   - a. Adding 2 to a number
   - b. Multiplying a number by 5
   - c. Dividing a number by 3
   - d. Subtracting 7 from a number

You know that $2 + 3 = 5$ is a true sentence, so you may regard both $2 + 3$ and 5 as names for the same number. You can think of 5 as a *standard name* of the number and of $2 + 3$ as a *number phrase*; however, in this unit we shall refer to both 5 and $2 + 3$, as well as such other collections of symbols as $2(3 + 5)$, $4 \times 5 + 2$, $4 - 2$, etc., as number phrases. Expressions like x, y, $x + 7$, $5x - 3$, and $5(x - 3) + 2$ that contain a variable will be referred to as *open phrases*.

An open phrase can be produced by following a set of instructions. Consider these instructions: "Take a number," "multiply the number by two," and "add three to the result." A slightly shortened version of these instructions and the resulting open phrases are displayed below.

INSTRUCTION	OPEN PHRASE
Take a number.	x
Multiply by two.	$2x$
Add three.	$2x + 3$

Not only can we build a phrase such as $2x + 3$, but we can also "undo" it. We expect the result to be x when the undoing is completed. (Since x is a variable, it represents some number. For this reason we treat x like a number in carrying out operations.) The display below shows how to undo the open phrase $2x + 3$.

INSTRUCTION	OPEN PHRASE
Take the open phrase.	$2x + 3$
Subtract three.	$2x$
Divide by two.	x

Notice that to undo *addition of a number*, we *subtract an equal number*. In the example above this means subtracting 3 from $2x + 3$. The display shows that $(2x + 3) - 3 = 2x$.

Also notice that to *undo multiplication by a number* we *divide by an equal number*. In the example above this means dividing $2x$ by 2. The display shows that $2x \div 2 = x$. (We can also write this in the form $\frac{2x}{2} = x$.) We can undo multiplication by any number except multiplication by 0. We cannot undo multiplication by 0 because it is not possible to divide by 0.

Now consider a set of instructions that can be used to produce a different open phrase: "Take a number," "divide the number by three," "subtract four from the result." The step-by-step building procedure looks like this:

INSTRUCTION	OPEN PHRASE
Take a number.	x
Divide by three.	$\frac{x}{3}$
Subtract four.	$\frac{x}{3} - 4$

Displayed below is a step-by-step procedure for undoing the open phrase $\frac{x}{3} - 4$.

INSTRUCTION	OPEN PHRASE
Take the open phrase.	$\frac{x}{3} - 4$
Add four.	$\frac{x}{3}$
Multiply by three.	x

The undoing procedure in this case is a little more complicated than in the previous example—but only a little. If you look at the display you can see that $\frac{x}{3}$ was obtained from $\frac{x}{3} - 4$ by adding 4 to $\frac{x}{3} - 4$. Why does this work? It is easy to see that if 2 is added to $3 - 2$, the result is 3. That is, $(3 - 2) + 2 = 3$. In the same way, $\left(\frac{x}{3} - 4\right) + 4 = \frac{x}{3}$. We conclude that to *undo subtraction of a number,* we *add an equal number.*

We are now half finished. We still need to explain how x is obtained from $\frac{x}{3}$. (Recall that the symbol $\frac{x}{3}$ means x divided by 3.) The instructions in the display indicate that $3 \cdot \left(\frac{x}{3}\right) = x$. That is, $\frac{x}{3}$ may be regarded as a number which when multiplied by 3 gives us the result x. In other words, to undo *division by a number* we *multiply by an equal number.* It is understood, of course, that division by 0 is not allowed.

Class Discussion 2b

1. The four basic operations are addition, multiplication, subtraction, and division.
   a. In building the phrase $2x + 3$, what operations were used?
   b. What undoing operations were used to obtain x from the open phrase $\frac{x}{3} - 4$?
2. An operation and the corresponding undoing operation can be shown in a single table. In the table below read the column on the left *down* and the column on the right *up*. Complete the instructions in the table.

INSTRUCTIONS FOR DOING	OPEN PHRASE	INSTRUCTIONS FOR UNDOING
Take a number.	x	______________by five.
______________by five.	$5x$	______________three.
______________three.	$5x + 3$	Take the open phrase.

3. Suppose that in building the open phrases shown below the instructions for each were to start with "Take a number...." What are the complete instructions that result in the open phrases that are listed?
   a. $2x - 7$
   b. $4y + 2$
   c. $5w - 4$

Summary—2

In this section you learned that an open phrase can be built by following a set of instructions. The resulting open phrase can be undone by reversing the order of the operations followed in the building procedure and replacing each by its corresponding undoing or inverse operation. Thus, subtraction can be used to undo addition. Addition can be used to undo subtraction. Division can be used to undo multiplication. And multiplication can be used to undo division.

Exercises—2

1. Describe an undoing operation for each of the following, provided that such an operation is possible.
   - a. Coming to school
   - b. Waking up
   - c. Baking bread
   - d. Turning on the TV set
   - e. Memorizing a poem
2. Complete the table below.

INSTRUCTION	OPEN PHRASE
Take a number.	
Multiply by four.	
Add five.	

3. Build a table like the one in exercise 2 for each of the following sets of instructions.
   - a. Take a number, multiply the number by five, and add ten to the result.
   - b. Take a number, divide the number by two, and subtract seven from the result.
   - c. Take a number, multiply the number by six, and add four to the result.

4. Build a table like the one in exercise 2 for each of the following sets of instructions.
   a. Take a number, multiply the number by seven, and add five to the product.
   b. Take a number, divide the number by six, and add four to the quotient.
   c. Multiply a number by five and add three to the product.

5. For each of the following open phrases write a suitable set of instructions that would, if followed, result in the given open phrase.
   a. $3x - 8$
   b. $2x + 14$
   c. $x - 5$
   d. $5x + 1$
   e. $4x$

6. Complete the table below for undoing an open phrase.

INSTRUCTION	OPEN PHRASE
Take the open phrase.	$7x - 3$
______________three.	$7x$
______________seven.	x

7. For each open phrase listed below build a table similar to the one shown in exercise 6 for undoing the given open phrase.
   a. $2x - 9$
   b. $x + 4$
   c. $3y + 5$
   d. $\frac{r}{2} + 10$

8. For each of the open phrases given below, build a table that lists the instructions for building the given open phrase and for undoing it.
   a. $5x + 3$
   b. $2x - 1$
   c. $4x + 7$
   d. $\frac{x}{7} + 5$

9. For each of the following descriptions, write an open phrase that corresponds to the given description.
   a. The sum of a number and seven
   b. Three less than two times a number
   c. Five more than three times a number
   d. The sum of a number and twice the number
   e. A number added to five
   f. Six less than four times a number
   g. The sum of a number, twice the number, and four
   h. Four less than seven times a number

3 Finding a Number

It is possible to describe a number in terms of a condition or set of conditions that the number satisfies without actually identifying the number. In cases like this it usually requires a little thinking to determine what the number is.

Class Discussion 3

1. Find the number that is referred to in each of the following situations.
   - **a.** "How old are you?" Tom asked Donovan.
     After a short pause, Donovan replied, "If you double my age and add six, the result is thirty-eight."
     How old is Donovan?
   - **b.** "I am thinking of a number," said Mary. "If you multiply this number by four and add three to the product, the result is twenty-seven."
     What is the number that Mary has in mind?
   - **c.** "What is the cost of an apple?" Dick asked the grocer.
     "If you buy three apples and give me a quarter, I will return four cents in change," replied the grocer.
     How much does one apple cost?
2. What number is referred to in each statement below?
   - **a.** If a number is multiplied by three and four is added to the product, the result is thirteen.
   - **b.** If a number is multiplied by two and five is subtracted from the product, the result is nine.
   - **c.** Twice a number is five more than seven.
   - **d.** If a number is doubled and six is subtracted from the product, the result is eight.
3. The statement in exercise 2a can be thought of as a set of instructions for writing an open sentence. The following interpretation is illustrative.

Take a number.	x
Multiply by three.	$3x$
Add four to the product.	$3x + 4$
The result is thirteen.	$3x + 4 = 13$.

The first three instructions give us the open phrase $3x + 4$. The last instruction suggests how to complete the open sentence $3x + 4 = 13$. Write open sentences for the remaining statements in exercise 2.

4. For each of the following descriptions make up a problem that involves the number indicated.
  a. Your age
  b. The total number of brothers and sisters in your family including yourself
  c. The number of years you have lived in your present home

5. Share the problems that you made up in exercise 4 with members of your class and let them try to find the number in each case.

Exercises—3

1. Find the number referred to in each of the following statements.
  a. When four is added to a number, the sum is nine.
  b. If seven is subtracted from a number, the result is eight.
  c. If ten is added to a number, the result is seventeen.
  d. A certain number minus six is nineteen.
  e. Nine less than a certain number is twenty-three.
  f. Four plus a certain number is fourteen.

2. Find the number referred to in each of the following statements.
  a. If a number is multiplied by seven and four is added to the product, the result is eighteen.
  b. If a number is doubled and five is subtracted from the product, the result is nine.
  c. The difference of four times a number and seven is thirteen.
  d. If a number is divided by three and two is subtracted from the quotient, the result is one.
  e. The sum of six times a number and five is twenty-three.

- **f.** The sum of seven and two times a number is twenty-nine.
- **g.** If a number is multiplied by four and eleven is added to the product, the result is thirty-five.

3. For each of the following open sentences, write a verbal statement similar to those in exercise 2.

a. $2x + 7 = 15$. **c.** $5x + 4 = 19$.
b. $\frac{x}{3} + 4 = 8$. **d.** $7x - 3 = 18$.

4. Find the solution for each open sentence listed in exercise 3.

5. For each number listed below, write a verbal statement in which the given number is treated as unknown.

a. 7 (*Hint:* The sum of three and two times the number is 17.)
b. 5 **c.** 2 **d.** 9

6. Translate each verbal statement that you wrote in exercise 5 into an open sentence. Give the list of verbal statements that you prepared in exercise 5 to another student and ask this student to write an open sentence for each. Check the open sentences that your classmate writes with your own.

7. Find the number referred to in each of the following situations.

- **a.** A number is added to itself. Four is subtracted from this sum. The result is twenty.
- **b.** A certain number plus three times this number is twenty-eight.
- **c.** Thirteen is added to two times a number. The result is forty-five.
- **d.** If five is subtracted from four times a number, the result is twenty-seven.
- **e.** If a number is divided by two, and three is added to the quotient, the result is nine.
- **f.** A number is divided by four. The result is eleven.
- **g.** A number is divided by five. Three is subtracted from the quotient. The result is two.
- **h.** A number is divided by four, and three is added to the quotient. The result is seven.
- **i.** If a number is divided by three, and four is added to the quotient, the result is nine.

j. A number is multiplied by seven. Six is subtracted from the product. The result is twenty-two.

8. The information in exercise 7a can be expressed by the open sentence $(x + x) - 4 = 20$. Write an open sentence for each of exercises 7b through 7j.

9. The open sentence $2x + 3 = 7$ can be translated into a verbal statement like the following:

"If a number is multiplied by two and three is added to the product, the result is seven."

Write a verbal statement for each of the following open sentences.

a. $2x - 3 = 11.$ **d.** $\frac{x}{2} - 7 = 10.$ **f.** $\frac{x}{3} + 5 = 9.$
b. $\frac{x}{4} + 5 = 8.$ **e.** $(x + 2x) - 4 = 5.$ **g.** $2x + 11 = 17.$
c. $5x + 2x = 28.$

10. Find the solution of each open sentence listed in exercise 9.

11. Express the information in each of the following situations with an open sentence.

a. Bill lives x miles from school. If the number of miles is tripled and three is added, the result is twenty-one.

b. Candy costs x cents per pound. After paying for two pounds of candy with a dollar bill, Mary received twelve cents in change.

c. The perimeter of a square is always four times the length of a side. The perimeter of a certain square plus seven is twenty-three.

4 A New Way to Find Solutions of Open Sentences

In the last section you learned that certain situations involving numbers can be expressed by writing mathematical sentences. If a situation involving numbers is concerned with a number that is unknown, then the mathematical sentence that we write will usually be an open sentence. In such cases we are interested in determining what the unknown number is, or what the solution of the open sentence is. In an open sentence like

$$3x = 12,$$

or like

$$x - 7 = 13,$$

it is easy to find the solution by inspection. However, if the open sentence is of the form

$$2x + 11 = 17,$$

or of the form

$$\frac{x}{5} - 6 = 4,$$

then the solution may be more difficult to find. In the open sentences

$$\frac{3x}{4} - 12 = 9$$

and

$$\frac{3x - 17}{4} = 48 - 2x$$

it is by no means obvious what the solutions are, and it may take considerable guessing to find them.

By using certain mathematical methods it is possible to find solutions of open sentences more quickly and efficiently than by having to resort to guessing. One purpose of this unit is to help you learn about such mathematical methods and how to use them in finding solutions of open sentences. If you complete this unit you should be able to use these methods to find solutions of open sentences like the last two displayed above. As you continue your study of mathematics, you will learn to find solutions of open sentences that look even more complicated than these.

As a first step in our search for mathematical methods that can be used to find solutions of open sentences, we need to look more carefully at mathematical sentences in general. In the discussion that follows we shall make use of the mathematical sentences listed below.

(a) $5 + 2 = 7$.
(b) $6 - 2 = 3$.
(c) $2x + 7 = 11$.
(d) $3(x + 2) = 3x + 6$.
(e) $x^2 - 2x = 0$.
(f) $x^2 + 13 - 7x = 1$.
(g) $0 \cdot x = 6$.

Sentence (a) is a true sentence, and sentence (b) is a false sentence. Each of sentences (c) through (g) is an open sentence. An open sentence is neither true nor false; however, if the variable is replaced by a number, then the resulting sentence will be true or false. Notice the symbol x^2 in sentences (e) and (f). This symbol means $x \cdot x$. If the x in x^2 is replaced by a number, this number is to be multiplied by itself.

In sentence (e) above, if x is replaced by 3, the resulting sentence is false. If x is replaced by 2, the resulting sentence is true. Therefore, 2 is a solution of the open sentence $x^2 - 2x = 0$. Do you think $x^2 - 2x = 0$ has any other solutions? If we replace x by 0 we get the sentence $0^2 - 2(0) = 0$, which is true. So we know that 0 is also a solution of $x^2 - 2x = 0$. It can be proved, although we will not do it here, that 2 and 0 are the *only* solutions of $x^2 - 2x = 0$. The collection of all solutions of an open sentence is called the *solution set* of the open sentence. Hence we write: *The solution set of* $x^2 - 2x = 0$ is $\{0, 2\}$. The symbol $\{0, 2\}$ is read "The set whose members are 0 and 2." The braces $\{ \}$ indicate that we are talking about a set.

How many numbers would you list within braces if an open sentence had exactly one solution? More than two solutions? No solution? Examine open sentences (c), (d), and (g) above to help you answer these questions.

Class Discussion 4a

1. In the open sentence $2x + 7 = 11$, replace x by 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. How many of the resulting sentences are true? How many members do you think there are in the solution set of $2x + 7 = 11$?

2. Suppose that x is replaced by 1 in the open sentence $3(x + 2) = 3x + 6$. Will the resulting sentence be true? (Since x appears in two places, x must be replaced by 1 in both places.) Answer the same question after replacing x by whole numbers other than 1.

3. How many whole numbers do you think there are in the solution set of $3(x + 2) = 3x + 6$?
4. Suppose that the whole numbers 0 through 10 are used as replacements for x in $x^2 + 13 - 7x = 1$. Which replacements, if any, result in true sentences?
5. Are there whole numbers greater than 10 that belong to the solution set of $x^2 + 13 - 7x = 1$? Give a reason for your answer.
6. How many whole numbers do you think there are in the solution set of $x^2 + 13 - 7x = 1$?
7. Suppose that x is replaced by 1 in $0 \cdot x = 6$. Will the resulting sentence be true?
8. Answer the question in exercise 7 after using whole numbers other than 1 as replacements for x.
9. How many whole numbers do you think there are in the solution set of $0 \cdot x = 6$?
10. Does every open sentence have a solution? Explain.

Summary—4a

The examples of open sentences considered in Class Discussion 4a indicate that the solution set of an open sentence may contain one or more members, or no member at all.

In particular, we found that the solution set of $2x + 7 = 11$ contains exactly one member; that the solution set of $x^2 + 13 - 7x = 1$ contains two members; that the solution set of $3(x + 2) = 3x + 6$ contains all whole numbers; and that the solution set of $0 \cdot x = 6$ contains no members. If a set contains no members, it is called the *empty set*. To symbolize the empty set, we simply write $\{ \}$. The conclusions stated above are summarized in the table below.

OPEN SENTENCE	SOLUTION SET	NUMBER OF MEMBERS IN THE SOLUTION SET
(c) $2x + 7 = 11$.	$\{2\}$	1
(d) $3(x + 2) = 3x + 6$.	$\{0, 1, 2, \ldots\}$	Infinitely many
(f) $x^2 + 13 - 7x = 1$.	$\{3, 4\}$	2
(g) $0 \cdot x = 6$.	$\{\ \}$	None

Note how the solution set of the open sentence $3(x + 2) = 3x + 6$ is represented. Since we have restricted ourselves to using only whole numbers in making replacements for the variable, we have listed only whole numbers in the solution set. The three dots in $\{0, 1, 2, \ldots\}$ indicate that the listing can be continued indefinitely.

Exercises—4a

1. For each open sentence below, list the whole numbers that are members of the solution set.
   - **a.** $2x = 3$.
   - **b.** $3x - 2 = 7$.
   - **c.** $n + 6 = n + 10$.
   - **d.** $y^2 + 6 - 5y = 0$.
   - **e.** $0 \cdot x = 0$.
   - **f.** $n^2 + 25 - 9n = 5$.
   - **g.** $\frac{x}{5} + 5 = 7$.
   - **h.** $0 + 6 = n$.
   - **i.** $0 - 6 = n$.
   - **j.** $0 \cdot 6 = n$.
   - **k.** $6 \div 0 = n$.
   - **l.** $0 \div 6 = n$.
   - **m.** $4(x + 3) = 4x + 12$.
2. For which open sentences in exercise 1 is the solution set empty?
3. For which open sentences in exercise 1 does the solution set contain all whole numbers?

Before proceeding, we call your attention to three open sentences included in the above set of exercises that deserve special mention. The open sentence $4(x + 3) = 4x + 12$ in exercise 1m illustrates an application of the distributive property of multiplication over addition. The open sentence $0 \cdot x = 0$ in exercise 1e is really a symbolic statement of the multiplication property of zero. According to this property *the product of any whole number and zero is zero.*

Finally, we direct your attention to the open sentence $6 \div 0 = n$

in exercise 1k. Recall that the solution set of this open sentence is empty. The reason is that there is no number which is the quotient of 6 and 0. If there were, the product of this number and 0 would have to be 6, and this is contrary to the multiplication property of zero.

Class Discussion **4b**

As stated earlier, one purpose of this unit is to help you learn about certain mathematical methods that can be used to find solutions of open sentences. In the remainder of this unit you will have an opportunity to explore these methods and to learn how to use them. In the beginning you will use these methods to find solutions of easy open sentences. Later on you will use the same methods to find solutions of harder open sentences such as the one below.

$$\frac{5x - 18}{3} = 92 - 3x.$$

(In this unit we shall not investigate special methods of finding solutions of open sentences that contain a variable multiplied by itself—that is, sentences which contain a symbol like x^2.)

In what follows we shall be concerned only with open sentences that express equalities. (Such open sentences are also called equations; see page 1.) To simplify discussion, we shall refer to the expression on the left of the equals symbol (=) as the *left side* of the open sentence, and to the expression on the right as the *right side*.

Let us explore the possibility of using addition and subtraction to obtain one open sentence from another with the result that both open sentences have the same solution set.

1. Look at the list of open sentences below.

(1) $x + 3 = 5$.
(2) $x + 4 = 6$.
(3) $x + 9 = 11$.
(4) $x + 12 = 15$.
(5) $x + 103 = 105$.
(6) $x = 2$.

a. Is 2 a solution of each open sentence in the list? Check your reply by replacing the variable in each open sentence by 2. Do you get a true sentence in each case?

b. Which open sentences in the list do not result in true sentences when you replace the variable by 2?

c. Which open sentences in the list have the same solution set?

d. If you add 1 to each side of $x + 3 = 5$, what open sentence do you obtain? Does the open sentence that you get have the same solution set as $x + 3 = 5$? How can you get the open sentence $x + 9 = 11$ from $x + 3 = 5$? From $x + 4 = 6$? From $x = 2$?

e. How can you get the open sentence $x + 103 = 105$ from $x + 3 = 5$? From $x + 4 = 6$? From $x + 9 = 11$? From $x = 2$?

f. Do you think adding the same number to each side of a given open sentence results in an open sentence that has the same solution set as the given open sentence? Explain your answer.

g. Complete the open sentences in the list below so that each will have the same solution set as $2x + 1 = 7$.

(1) $2x + 3 =$ ____.
(2) $2x + 4 =$ ____.
(3) $2x +$ ____ $= 11$.
(4) $2x + 0 =$ ____.

2. Consider the open sentence $x + 12 = 21$. Its solution set is $\{9\}$.

a. Which open sentences in the list below have the same solution set?

(1) $x + 11 = 20$.
(2) $x + 9 = 18$.
(3) $x + 7 = 15$.
(4) $x + 1 = 10$.
(5) $x + 0 = 9$.

b. Which open sentences in exercise 2a do not have the same solution set as $x + 12 = 21$?

c. If you subtract 1 from each side of $x + 12 = 21$, what open sentence do you obtain? Does the open sentence that you get have the same solution set as $x + 12 = 21$?

d. How can you get the open sentence $x + 1 = 10$ from $x + 9 = 18$? From $x + 11 = 20$?

e. How can you get the open sentence $x + 0 = 9$ from $x + 1 = 10$? From $x + 9 = 18$? From $x + 11 = 20$?

f. Do you think subtracting the same number from each side of an open sentence results in an open sentence that has the same solution set as the original open sentence? Explain your answer.

g. Consider the open sentence $x + 7 = 4$. If we work only with whole numbers, can you subtract 7 from both sides? What is the greatest whole number that you can subtract from both sides? Does this open sentence have a solution that is a whole number? Explain your answer.

h. Complete the open sentences in the list below so that all will have the same solution set as $4x + 7 = 19$.

(1) $4x + \_\_\_ = 12$.
(2) $4x + 4 = \_\_\_$.
(3) $4x + 1 = \_\_\_$.

The explorations in exercise 1 above suggest that if you add the same number to both sides of an open sentence the resulting open sentence will have the same solution set as the open sentence with which you started.

In view of exercise 2 above, a similar observation can be made about subtraction of the same number from both sides of an open sentence. It is understood, of course, that in subtracting one whole number from another, we cannot subtract a greater number from a lesser number.

Class Discussion **4c**

Now let us combine the ideas introduced in Class Discussion 4b with the "undoing" idea described in Section 2 and use them both to solve certain open sentences. (To *solve* an open sentence means to find its solution set.)

Let us start by considering again a question that should be familiar to you from the work in Section 2. What is a set of instructions that can be used to undo the open phrase $x + 7$?

INSTRUCTION	OPEN PHRASE
Take the open phrase.	$x + 7$
Subtract 7.	x

Now suppose that we apply the same kind of undoing instructions to the open sentence $x + 7 = 13$. What will be the result?

INSTRUCTION	OPEN SENTENCE
Take the open sentence.	$x + 7 = 13.$
Subtract 7 from each side.	$x = 6.$

If you replace x by 6 in $x = 6$, the resulting sentence is $6 = 6$, which is a true sentence. Therefore, 6 is a solution of $x = 6$. Furthermore, it is clear that 6 is the only solution of $x = 6$. Hence, the solution set of $x = 6$ is $\{6\}$. Now, the all-important question is this: *Is* $\{6\}$ *also the solution set of* $x + 7 = 13$*?* The answer is "yes." We can explain why in the following way:

> If $x + 7 = 13$ has a solution, there is a replacement for x that makes $x + 7 = 13$ a true sentence. Assume that this replacement has been made. Then, $x + 7$ and 13 represent the same number.
>
> So, if we subtract 7 from both sides of $x + 7 = 13$, we are really subtracting 7 from the same number in both cases. So the differences will be the same. On the left side the difference is represented by x and on the right side by 6. So x and 6 represent the same number. That is, $x = 6$.
>
> Thus we know that if $x + 7 = 13$ has a solution, it is a solution of $x = 6$. But the solution set of $x = 6$ contains only the number 6. So 6 is the only possible solution that $x + 7 = 13$ can have.
>
> If we replace x by 6 in $x + 7 = 13$, we get $6 + 7 = 13$, which is a true sentence. We conclude: The solution set of $x + 7 = 13$ is $\{6\}$.

The explanation above is really a proof that $x + 7 = 13$ has the same solution set as $x = 6$. In fact, we have proved a special case of the following general statement:

> *Subtraction property of equality for open sentences:* If an open sentence expresses an equality and we subtract the same number from both sides, we get another open sentence that has the same solution set as the original open sentence. (This property holds whenever subtraction is possible. See exercise 2g in Class Discussion 4b.)

1. A similar property can be stated for addition. Only a few words have to be changed in the statement above. State the *addition property of equality for open sentences* in your own words.
2. Below is an example which illustrates the use of an "undoing" procedure which involves addition.

INSTRUCTION	OPEN SENTENCE
Take the open sentence.	$x - 5 = 11.$
Add 5 to each side.	$x = 16.$

   a. Note that we have added 5 to each side of $x - 5 = 11$. Why did we add 5 rather than some other number?
   b. What is the solution set of $x = 16$?
   c. What is the solution set of $x - 5 = 11$?
   d. Why is the solution set of $x - 5 = 11$ the same as the solution set of $x = 16$?

A proof that $x - 5 = 11$ and $x = 16$ have the same solution set would proceed in the same way as the proof on page 21. From now on, however, we are not going to stop and prove that every undoing procedure that we use is legal. Instead, we shall rely on properties of equality for open sentences to justify the use of such procedures. We can do this because the properties themselves can be proved.

So far we have considered the addition and subtraction properties of equality for open sentences. You may anticipate that there are similar properties for multiplication and division, but before looking at these let us concentrate on the use of the addition and subtraction properties.

To reduce the amount of writing in using undoing procedures, we shall introduce some shorthand. For example, the instruction "Add 7 to each side" will be symbolized by A_7, and the instruction "Subtract 3 from each side" will be symbolized by S_3. In examples (1) and (2) below we use this shorthand notation.

EXAMPLE (1): Find the solution set of $y - 7 = 4$.

$$y - 7 = 4.$$

$$A_7 \qquad y = 11.$$

The solution set is {11}.

Example (2): Find the solution set of $x + 3 = 9$.

$x + 3 = 9.$

S$_3$ $\quad x \quad = 6.$

The solution set is $\{6\}$.

Exercises—4c

1. Use the methods you have learned to find the solution set of each of the following open sentences.
   - a. $t + 5 = 7.$
   - b. $m + 4 = 14.$
   - c. $x - 7 = 17.$
   - d. $s + 7 = 10.$
   - e. $c - 5 = 8.$
   - f. $y + 11 = 11.$
   - g. $d - 2 = 19.$
   - h. $x + 4 = 9.$
   - i. $y - 3 = 0.$
   - j. $22 + x = 31.$
   - k. $13 + m = 21.$
   - l. $y - 7 = 0.$
2. Find the solution set of each of the following open sentences.
   - a. $m + 9 = 35.$
   - b. $w - 3 = 17.$
   - c. $x - 11 = 0.$
   - d. $16 + m = 27.$
   - e. $x + 27 = 15.$
   - f. $x + 7 = 2.$
   - g. $y - 4 = 0.$
   - h. $x + 17 = 8.$
   - i. $m - 2 = 0.$
   - j. $w + 5 = 25.$
3. Since only whole numbers are being used as replacements for a variable, for which open sentences in exercise 2 is the solution set empty?
4. Use the methods described in this section to solve the open sentences listed below. In each exercise, state the property on which the procedure that you use depends.
   - a. $z + 3 = 13.$
   - b. $x - 8 = 21.$
   - c. $x + 38 = 38.$
   - d. $m - 7 = 18.$
   - e. $z + 4 = 22.$

Class Discussion **4d**

John came to class one day with a problem. He said, "I can use the methods we have learned to find the solution set of an open sentence like $x + 7 = 12$, and like $x - 10 = 7$. But the methods we have learned do not work in finding the solution set of an open sentence like $4x = 24$. If I add 4 to each side of $4x = 24$, I get $4x + 4 = 28$; and if I subtract 4 from each side I get $4x - 4 = 20$. Neither of these open sentences is as simple as the one with which I started. How can I get an open sentence with x rather than $4x$ on one side, and which has the same solution set as $4x = 24$?"

"Undo the multiplication by 4 on each side of the open sentence $4x = 24$," replied Chris. "To undo multiplication by a number you divide by an equal number. If you divide each side by 4 the result on the left side is x, and the result on the right side is 6. In other words, you get the open sentence $x = 6$."

"The open sentence $x = 6$ has the form that I want," said John, "and I can see that its solution set is $\{6\}$. Does $4x = 24$ have the same solution set?"

"Yes," said Chris. "Because $4 \times 6 = 24$, and 4 times any other number is not 24."

---

1. Consider the open sentence $12x = 48$. If you divide each side by 12, what open sentence do you get? What is the solution set of the open sentence that you get? Is this also the solution set of $12x = 48$?
2. Divide each side of $12x = 48$ in turn by 1, 2, 3, 4, and 6. List the open sentences that you obtain. What is the solution set of each?
3. Is it possible to divide both sides of the open sentence $12x = 48$ by whole numbers other than 1, 2, 3, 4, 6, and 12?

4. Do you think dividing both sides of an open sentence by the same non-zero number results in an open sentence that has the same solution set as the original open sentence? Are there any restrictions on what the divisors can be if we work only with whole numbers?

The exercises above and the preceding discussion suggest that division on both sides of an open sentence that expresses an equality has the following property.

> *Division Property of Equality for Open Sentences:* If an open sentence expresses an equality and we divide both sides by the same non-zero number, we get another open sentence that has the same solution set as the original open sentence. (This property holds whenever division is possible. Its use in the present unit is limited, because the quotient of two whole numbers is not always a whole number.)

So far you have learned about three properties of equality for open sentences—one for addition, one for subtraction, and one for division. It is natural to suppose that there is also a *multiplication property of equality for open sentences.* How shall we state this property? The exercises that follow will help you understand what is involved.

5. Study the four open sentences in the list below.

(1) $x = 8$.
(2) $2x = 16$.
(3) $6x = 48$.
(4) $18x = 144$.

- **a.** What is the solution set of each open sentence?
- **b.** Do all open sentences in the list have the same solution set?
- **c.** How can you get the open sentence $18x = 144$ from $6x = 48$? From $2x = 16$? From $x = 8$?

6. Do you think that multiplying by the same number on both sides of an open sentence results in an open sentence that has the same solution set as the original open sentence?

7. Let us multiply each side of $2x = 16$ by zero. A record of the work is shown below.

$$\begin{aligned} 2x &= 16. \\ 0\cdot(2x) &= 0\cdot 16. \\ (0\cdot 2)x &= 0. \\ 0\cdot x &= 0. \end{aligned}$$

a. What is the solution set of $2x = 16$?
b. What is the solution set of $0\cdot x = 0$?
c. Do the open sentences $2x = 16$ and $0\cdot x = 0$ have the same solution set?
d. Do you want to revise your answer to exercise 6?

Below is a statement of the property we have been exploring. If you do not understand it completely, look again at exercises 5 through 7.

> *Multiplication property of equality for open sentences:* If an open sentence expresses an equality and we multiply both sides by the same non-zero number, we get another open sentence that has the same solution set as the original open sentence.

Class Discussion **4e**

In Exercises—4c we concentrated on solving open sentences in which we added to undo subtraction or in which we subtracted to undo addition. In using these procedures we depended on the addition and subtraction properties of equality for open sentences.

Let us now give attention to solving open sentences in which the undoing procedures involve multiplication and division.

1. From the discussion at the beginning of Class Discussion 4d you know that an open sentence like $3x = 15$ can be solved by undoing multiplication. In this case you divide both sides by 3. The procedure is shown below.

$$\begin{aligned} & 3x = 15. \\ D_3 \qquad & x = 5. \end{aligned}$$

The solution set of $3x = 15$ is $\{5\}$.

Note that we have used the symbol D_3 in place of the instruction "Divide both sides by 3." What property assures us that $x = 5$ and $3x = 15$ have the same solution set?

2. The solution set of $\frac{x}{2} = 13$ can be found by the procedure shown below. The symbol M_2 means "Multiply both sides by 2."

$$\frac{x}{2} = 13.$$

M_2 $\qquad x = 26.$

The solution set of $\frac{x}{2} = 13$ is $\{26\}$.

- **a.** Let us trace the reasoning involved in solving the open sentence $\frac{x}{2} = 13$. Since $\frac{x}{2}$ means x *divided by 2*, we must undo division by 2 to solve this open sentence. To undo the division, we must multiply both sides by 2, like this: $2 \cdot \left(\frac{x}{2}\right) = 2 \cdot 13$. On the left we get $2 \cdot \left(\frac{x}{2}\right) = x$. What do we get on the right side?
- **b.** It is easy to see that $\{26\}$ is the solution set of $x = 26$. In view of the procedure that we used, $\frac{x}{2} = 13$ has the same solution set. What property enables us to say this?
- **c.** To satisfy yourself that 26 is a solution of $\frac{x}{2} = 13$ replace x by 26. Do you get a true sentence?
- **d.** The procedure in exercise 2c is referred to as *checking*. What is it that you checked in exercise 2c, your method of solving the open sentence or your computation?

Exercises—4e

1. Use the methods that you learned in Class Discussion 4e to find the solution set of each of the following open sentences. Check the solutions you obtain.

a. $5x = 70.$

b. $12n = 60.$

c. $\frac{x}{4} = 10.$

d. $\frac{x}{7} = 9.$

e. $3y = 0.$

f. $2m = 7.$

g. $\frac{x}{10} = 1.$

h. $7x = 98.$

i. $\frac{t}{2} = 0.$

j. $\frac{3x}{4} = 9.$

k. $\frac{5s}{7} = 10.$

l. $\frac{6s}{11} = 8.$

2. Since only whole numbers are being used as replacements for a variable, for which open sentences in exercise 1 is the solution set empty?
3. Use the methods you have learned to solve the open sentences listed below. In each exercise state the property (or properties) on which the procedures that you use depend.

   a. $9x = 108.$
   b. $x - 9 = 15.$
   c. $\frac{x}{7} = 3.$
   d. $x + 7 = 23.$
   e. $\frac{4x}{3} = 8.$

Class Discussion **4f**

1. An open sentence such as $5x + 3 = 28$ can be solved by first undoing addition and then undoing multiplication. A record of the work involved in solving this open sentence is shown below.

$$\begin{array}{lrl} & 5x + 3 &= 28. \\ S_3 & 5x &= 25. \\ D_5 & x &= 5. \end{array}$$

   The solution set of $5x + 3 = 28$ is $\{5\}$.

   a. In the record of the work shown above there is a list of three open sentences. Would you say that $5x = 25$ is a simpler open sentence than $5x + 3 = 28$? Would you say that $x = 5$ is the simplest open sentence in the list?
   b. Do all three open sentences in the list have $\{5\}$ as their solution set?
   c. What property assures us that $5x = 25$ has the same solution set as $5x + 3 = 28$?
   d. What property assures us that $x = 5$ has the same solution set as $5x = 25$?

2. Shown below is a record of the work involved in solving $\frac{x}{5} - 3 = 27$.

$\frac{x}{5} - 3 = 27.$
A_3 $\frac{x}{5} = 30.$
M_5 $x = 150.$
The solution set of $\frac{x}{5} - 3 = 27$ is $\{150\}$.

a. Look at the three open sentences in the list above. Which open sentence in this list do you consider the simplest? Does this open sentence have the same form as the one you pointed out as the simplest in the list of exercise 1? How would you describe this form?

b. Do all three open sentences in the list above have the same solution set?

c. What property assures us that $\frac{x}{5} = 30$ has the same solution set as $\frac{x}{5} - 3 = 27$?

d. What property assures us that $x = 150$ has the same solution set as $\frac{x}{5} = 30$?

3. Try to explain why $x = 150$ has the same solution set as $\frac{x}{5} - 3 = 27$.

The scheme we have developed for solving open sentences can be described in the following way. Starting with an open sentence that we want to solve, we carry out undoing procedures to get a list of simpler open sentences ending with one which has the form $x = a$, where a is some whole number. In exercise 2 the open sentence $\frac{x}{5} = 30$ is simpler than $\frac{x}{5} - 3 = 27$; and $x = 150$ is simpler than $\frac{x}{5} = 30$.

4. For each open sentence below, what undoing procedure should be used first to obtain a simpler open sentence that has the same solution set as the given open sentence? In each case, tell what property assures you that the open sentence you get has the same solution set as the given open sentence.

a. $x - 15 = 17.$
b. $3x - 5 = 1.$
c. $7x + 2 = 9.$
d. $\frac{2x}{3} + 5 = 7.$
e. $5x - 18 = 27.$

5. A second simplification can be attempted with some of the open sentences in exercise 4. In each case describe the undoing procedure and tell what property assures you that the open sentence you get has the same solution set as the one you got in the first simplification.

6. Solve the open sentence $\frac{3x}{4} - 12 = 9$. Begin by writing down a list of increasingly simpler open sentences that have the same solution set. The last open sentence in the list should be of the form $x = a$, where a is a whole number. What is the solution set of the given open sentence?

Exercises—4f

1. Use the methods that you have learned to find the solution set of each of the following open sentences.
   - **a.** $3x = 15.$
   - **b.** $2y = 18.$
   - **c.** $7z = 28.$
   - **d.** $\frac{x}{3} = 5.$
   - **e.** $\frac{m}{2} = 7.$
   - **f.** $11x = 121.$
   - **g.** $\frac{y}{7} = 3.$
   - **h.** $27m = 81.$

2. Solve each of the following open sentences. Check the solutions you obtain.
   - **a.** $3x - 7 = 5.$
   - **b.** $2x + 7 = 21.$
   - **c.** $2x - 8 = 30.$
   - **d.** $\frac{2x}{3} = 6.$
   - **e.** $5x - 17 = 13.$
   - **f.** $\frac{2m}{7} = 8.$
   - **g.** $8x - 3 = 5.$
   - **h.** $\frac{3x}{8} = 12.$
   - **i.** $17x - 11 = 40.$
   - **j.** $\frac{x}{3} - 5 = 0.$
   - **k.** $13x + 18 = 44.$
   - **l.** $\frac{5y}{6} = 10.$

3. Solve each of the following open sentences. Check the solutions you obtain.
   - **a.** $\frac{2x}{3} + 4 = 14.$
   - **b.** $\frac{3x}{4} - 7 = 14.$
   - **c.** $\frac{x}{3} - 5 = 2.$
   - **d.** $\frac{2m}{5} - 4 = 6.$
   - **e.** $8 + \frac{3y}{7} = 23.$
   - **f.** $3(n + 6) = 24.$
   - **g.** $\frac{3m + 7}{2} = 8.$
   - **h.** $4(x + 3) - 5 = 11.$
   - **i.** $\frac{3x - 17}{4} = 48 - 2x.$
   - **j.** $\frac{5x - 18}{3} = 92 - 3x$

5 Solving Puzzle Problems

The example that follows is an example of a puzzle problem. A problem like this can often be solved by first translating the problem into an open sentence. The solution set of the open sentence can be used to answer the question in the problem.

EXAMPLE: A history class has 33 students. There are 7 more girls than boys. How many boys and how many girls are there in the class?

Here is an account of the way in which this puzzle problem can be solved.

Translation:

Let x represent the number of boys.	x
Then $x + 7$ represents the number of girls.	$x + 7$
Either $x + (x + 7)$ or $2x + 7$ can be used to represent the sum of the number of boys and the number of girls.	$2x + 7$
The problem states that the number of students is 33.	33
And the sum of the number of boys and the number of girls is the same as the number of students.	$2x + 7 = 33.$

Solution:

$$\begin{array}{lr} & 2x + 7 = 33. \\ S_7 & 2x = 26. \\ D_2 & x = 13. \end{array}$$

The solution set of the open sentence is $\{13\}$.

The number of boys is 13.

The number of girls is $13 + 7$, or 20.

Class Discussion 5

Consider the following example of a puzzle problem:

EXAMPLE: The length of a rectangle is 4 inches more than its width. If the perimeter of the rectangle is 28 inches, what is its width?

The exercises that follow should help you translate the above puzzle problem into an open sentence and help you answer the question in the problem.

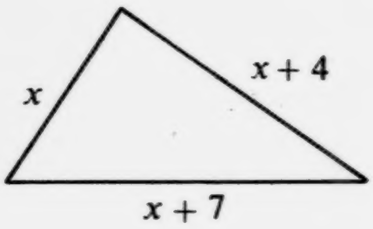
1. If we let x represent the number of inches in the width, what open phrase can be used to represent the number of inches in the length?
2. What open phrase can be used to represent the number of inches in the four sides of the rectangle?
3. What is the number of inches in the perimeter of the rectangle? (The answer to this question is given in the statement of the problem.)
4. Write the open sentence suggested by the answers to the questions in exercises 2 and 3.
5. Find the solution set of the open sentence.
6. Answer the question in the problem.

Exercises—5

For each puzzle problem stated below, write an open sentence that can be used to answer the question in the problem. Solve the open sentence, and answer the question in the problem.

1. If seven is added to three times a number, the sum is 46. What is the number?
2. The length of a rectangle is seven inches greater than the width. If the perimeter of the rectangle is 54 inches, what is the width?

3. If five is subtracted from four times a number, the result is 39. What is the number?
4. The sum of the ages of Dick and Dan is 42 years. If Dan is eight years older than Dick, how old is Dick?
5. In a certain election, 735 votes were cast. If Bob received 29 votes more than Jerry, how many votes did Bob receive?
6. A loaded truck weighs 12,800 pounds. This is 800 pounds more than twice the weight of the empty truck. What is the weight of the empty truck?
7. The lengths of the sides of a triangle are indicated by the open phrases shown in the diagram at the right. If the perimeter of the triangle is 53 inches, how long is the shortest side?



8. Don, Tim, and Jack pooled their money to start a small business. Don invested twice as much as Jack. Tim invested 16 dollars more than Jack. If the total amount invested by the three boys is 96 dollars, how much did Jack invest?

General Summary

Three kinds of mathematical sentences were presented in this unit—*true, false,* and *open.* A sentence like $2 + 2 = 4$ is called a true sentence; a sentence like $2 + 2 = 5$ is referred to as a false sentence; and a sentence like $2 + x = 4$ is called an open sentence. The last sentence is called "open" because it contains a variable which is open for replacement by numbers. A number that is used as a replacement for a variable in an open sentence and that makes the resulting sentence true is called a *solution* of the open sentence.

In this unit we used only whole numbers as replacements for a variable. As you continue your study of mathematics you will learn to use rational numbers and real numbers to obtain solutions of open sentences.

By using whole numbers as replacements for a variable in an open sentence, we found that some open sentences we encountered had exactly one solution; some had two solutions; some had infinitely many solutions; and some had no solution. In any case we referred to the collection of all solutions of a particular open sentence as the *solution set* of the open sentence.

To solve open sentences quickly and efficiently, we developed a systematic scheme of undoing operations. We used this scheme to obtain a short list of open sentences, all having the same solution set. Each open sentence in the list was simpler than the one that preceded it, and the simplest open sentence had the form $x = a$, where a is a whole number.

To justify the use of the various undoing procedures, we relied on addition, subtraction, multiplication, and division properties of equality for open sentences.

Review Exercises

1. For each sentence, state whether the given sentence is true, false, or open.

a. $y + 5 = 11$.
b. $7 - 2 = 5$.
c. $2 + 8 = 11$.
d. $5 - 2x = 3$.
e. $x + 2 = x + 7$.
f. $x^2 + 5 - x = 5$.
g. $2(x + 7) = 2x + 14$.
h. $3 + 0 = w$.

2. If only whole numbers are used as replacements for the variables in the open sentences in exercise 1, for which open sentences will the solution set contain—

a. Exactly one member?
b. No member?
c. More than one member?
d. All whole numbers?

3. Describe the undoing action (or operation) that is needed to undo each of the following.

a. Adding three to a number
b. Opening the door and picking up the newspaper

  - **c.** Multiplying a number by seven
  - **d.** Opening a book and underlining a sentence

4. Write an open phrase for each of the following sets of instructions.
  - **a.** Take a number; multiply the number by three; and add four to the product.
  - **b.** Take a number; divide the number by five; and subtract seven from the quotient.
  - **c.** Take a number; multiply the number by seven; and subtract three from the product.

5. Write an open phrase for each of the following descriptions.
  - **a.** Five less than three times a given number
  - **b.** The sum of a given number, twice the given number, and seven
  - **c.** Six more than the quotient obtained when a given number is divided by two

6. Find the solution set of each of the following open sentences.
  - **a.** $m + 11 = 24$.
  - **b.** $3x - 5 = 1$.
  - **c.** $y - 11 = 23$.
  - **d.** $17x - 11 = 40$.
  - **e.** $\frac{x}{3} - 3 = 2$.
  - **f.** $2x - 7 = 9$.
  - **g.** $\frac{x}{3} + 3 = 5$.
  - **h.** $\frac{2m}{5} - 3 = 7$.
  - **i.** $3x + 11 = 17$.
  - **j.** $2m - 8 = 6$.

7. For each of the following puzzle problems, write an open sentence that can be used to answer the question in the problem. Solve the open sentence, and answer the question in the problem.
  - **a.** The sum of a number and three times the number is 80. What is the number?
  - **b.** A board 56 inches long is cut into two pieces so that one piece is three times as long as the other. How long is the shorter piece?
  - **c.** Joan and Jim ran for class president. If 602 votes were cast and Jim got 50 votes more than Joan did, how many votes did Joan get?
  - **d.** Bill has 75 cents in nickels and dimes. He has twice as many dimes as nickels. How many dimes does he have?

4

Geometry

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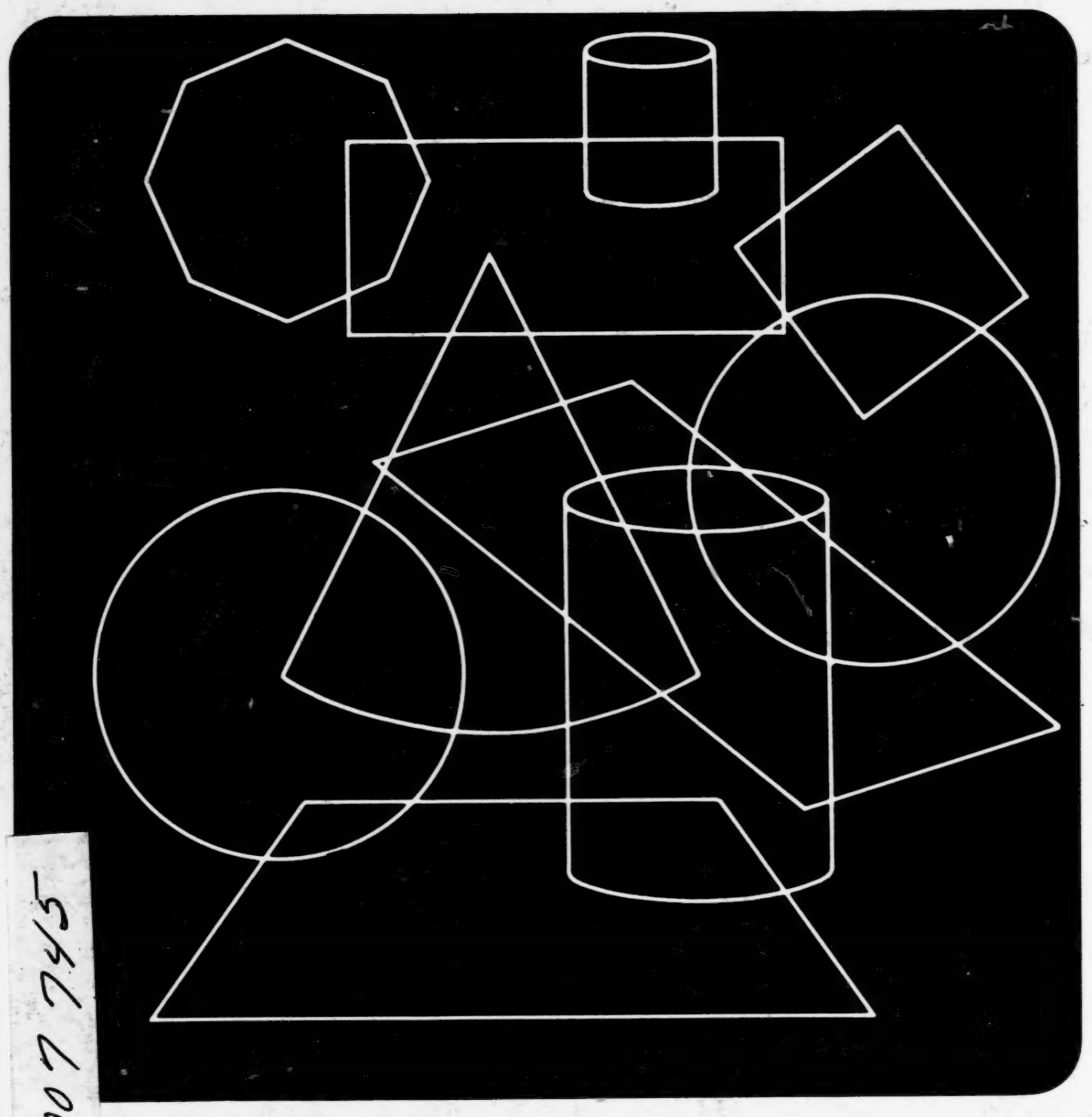

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UNIT FOUR OF

Experiences in Mathematical Discovery

Geometry

NATIONAL COUNCIL OF
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1201 Sixteenth Street, N.W., Washington, D.C. 20036

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Preface

"Experiences in Mathematical Discovery" is a series of ten self-contained units, each of which is designed for use by students of ninth-grade general mathematics. These booklets are the culmination of work undertaken as part of the General Mathematics Writing Project of the National Council of Teachers of Mathematics (NCTM).

The titles in the series are as follows:

Unit 1: *Formulas, Graphs, and Patterns*

Unit 2: *Properties of Operations with Numbers*

Unit 3: *Mathematical Sentences*

Unit 4: *Geometry*

Unit 5: *Arrangements and Selections*

Unit 6: *Mathematical Thinking*

Unit 7: *Rational Numbers*

Unit 8: *Ratios, Proportions, and Percent*

Unit 9: *Measurement*

Unit 10: *Positive and Negative Numbers*

This project is experimental. Teachers may use as many units as suit their purposes. Authors are encouraged to develop similar approaches to the topics treated here, and to other topics, since the aim of the NCTM in making these units available is to stimu-

late the development of special materials that can be effectively used with students of general mathematics.

Preliminary versions of the units were produced by a writing team that met at the University of Oregon during the summer of 1963. The units were subsequently tried out in ninth-grade general mathematics classes throughout the United States.

Oscar F. Schaaf, of the University of Oregon, was director of the 1963 summer writing team that produced the preliminary materials. The work of planning the content of the various units was undertaken by Thomas J. Hill, Oklahoma City Public Schools, Oklahoma City, Oklahoma; Paul S. Jorgensen, Carleton College, Northfield, Minnesota; Kenneth P. Kidd, University of Florida, Gainesville, Florida; and Max Peters, George W. Wingate High School, Brooklyn, New York.

The Advisory Committee for the General Mathematics Writing Project was composed of Emil J. Berger (*chairman*), St. Paul Public Schools, St. Paul, Minnesota; Irving Adler, North Bennington, Vermont; Stanley J. Bezuszka, S.J., Boston College, Chestnut Hill, Massachusetts; Eugene P. Smith, Wayne State University, Detroit, Michigan; and Max A. Sobel, Montclair State College, Upper Montclair, New Jersey.

The following people participated in the writing of either the preliminary or the revised versions of the various units:

Donald H. Firl, Rochester Public Schools, Rochester, Minnesota

Carroll E. Fogal, University of Florida, Gainesville, Florida

Edward F. Gottlieb, Madison High School, Portland, Oregon

Kenneth P. Kidd, University of Florida, Gainesville, Florida

Edna K. Lassar, Polytechnic Institute of Brooklyn, Brooklyn, New York

John F. LeBlanc, Racine Public Schools, Racine, Wisconsin

Scott D. McFadden, Spencer Butte Junior High School, Eugene, Oregon

Max Peters, George W. Wingate High School, Brooklyn, New York

Leonard Simon, Bureau of Curriculum Research, New York Public Schools, New York, New York

Irwin N. Sokol, Collinwood High School, Cleveland, Ohio

William H. Spragens, University of Louisville, Louisville, Kentucky

Mary E. Stine, Fairfax County Schools, Fairfax, Virginia

Arthur J. Wiebe, Pacific College, Fresno, California

John E. Yarnelle, Hanover College, Hanover, Indiana

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EMIL J. BERGER
Chairman, Advisory Committee
General Mathematics Writing Project

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Experiences in Mathematical Discovery

Geometry

1 Geometric Properties

In one of the myths of the ancient Greeks there was a king named Midas. Midas once helped a wood spirit who was a companion of a god. To reward Midas, the god offered to grant him any wish he chose. Midas, who was a miser, asked that everything he touched be turned into gold. He soon regretted that he had this magic touch, for he found that it affected food and drink as well as other things.

Class Discussion 1

An apple has many different properties, among which are its taste, color, weight, food value, size, shape, and position in space.

1. When Midas touched a red apple, which of these properties were changed?
2. Which remained unchanged?

Properties of an object that relate only to the size, shape, or position of the object or its parts are called *geometric properties*. Two objects that have different physical and chemical properties, such as a real apple and a gold apple, may have the same geometric properties.

Exercises—1

Each of the statements below describes a property of some object or objects. In each case, state whether the property is or is not a geometric property.

1. Grass is green.

2. The earth is shaped like a ball.

3. John weighs 125 pounds.

4. John is 5 feet 6 inches tall.

5. A stop sign is made of metal.

6. The outline of a stop sign has eight sides (Figure 1).

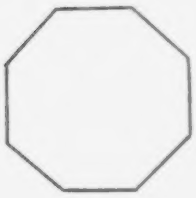


Fig. 1

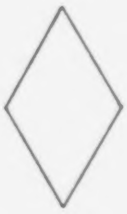


Fig. 2

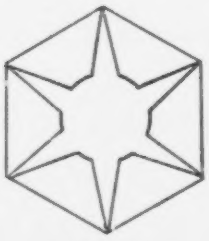


Fig. 3

7. The diamonds on playing cards are printed in red ink.

8. Each diamond on a playing card has four sides that have the same length (Figure 2).

9. A snowflake is a crystal of frozen water.

10. If the tips of the arms of a snowflake are joined in succession, as in Figure 3, a closed figure is formed that has six sides.

11. The top of a table is flat.

12. The mirror of a reflecting telescope is curved.

13. The edge of a yardstick is straight.

14. A certain rubber glove is yellow on the outside and white on the inside.

15. If a right-hand glove is turned inside out, it becomes a left-hand glove.

16. The edges of this page form a four-sided figure like that shown in Figure 4.

Fig. 4

2 Mathematical Models of Physical Space

Physical objects exist in physical space. Physical space is like a stage, and the physical objects in it are like the actors on the stage.

It is often useful to think of the geometric properties of an object apart from its other properties. To do so, we think of the geometric properties as properties of the portion of space that is occupied by the object. Thus, the geometric properties of an apple are the properties of the portion of space that is occupied by the apple. If the apple were changed to gold by the touch of Midas, it would still have the same geometric properties because it would still occupy the same portion of space.

To study and understand physical space, we try to form a mental picture of it. This mental picture is called *mathematical space*, or a *mathematical model* of physical space. To construct a mathematical model of physical space, mathematicians do three things:

(1) They put into the model some objects to match the objects found in physical space.

(2) They observe some properties of physical space and then assign these properties and some others to their model.

(3) From these assigned properties of the model, they figure out other properties of the model by reasoning.

Three different models of physical space have been constructed.

One was put into nearly complete form over two thousand years ago (about 300 B.C.) by the Greek mathematician Euclid, so it is known as the *Euclidean model*, or *Euclidean geometry*. A second model, constructed by the German mathematician Gauss (1777-1855), the Hungarian mathematician Bolyai (1802-1860), and the Russian mathematician Lobachevsky (1793-1856), is known as *hyperbolic geometry*. A third model, constructed by the German mathematician Riemann (1826-1866), is known as *elliptic geometry*. In this unit we deal only with the Euclidean model, the one that is commonly used in everyday life.

Class Discussion 2

In this discussion we examine objects in mathematical space that match objects found in physical space.

1. A speck of dust is so small, compared to its surroundings, that for many purposes we may neglect its size altogether. The mathematical object used to match or represent a physical object whose size may be neglected is called a *point*.

a. Does a point have length?
b. Does it have width?
c. Does it have thickness?
d. We usually represent a point by making a dot with a pencil or a piece of chalk. Is a dot made with a pencil or a piece of chalk a perfect representation of a point?

Mathematical space is considered to be a set of points, with certain relationships that exist among the points.

2. The path of a jet plane is made visible by the trail of ice crystals formed from its exhaust gases. If we neglect the width and thickness of the trail, the mathematical object that matches it is called a *mathematical line* (usually referred to simply as a line). To represent a line, we usually draw a pencil line or a chalk line, and we say that we "draw a line."

a. Draw a line that represents the path followed by the center of the earth as it moves around the sun.

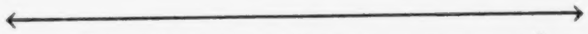
b. Draw a line that represents the path followed by the center of a fly ball hit by a batter in baseball.

c. Draw a line that represents the path of the center of a ball that is dropped, but not thrown, from your hand.

d. Draw a line that represents the crease made in a sheet of paper when you fold it once and press the crease flat.

e. Which of these lines are straight? Which are curved?

A straight line has no endpoints but extends indefinitely in opposite directions. To indicate this property of a line, we draw tips of arrows in the positions shown below when we "draw a line."



3. We often use a wire cheese cutter to cut a piece of cheese into two pieces. The path cut through the cheese by the wire becomes a boundary separating one piece of cheese from the other. The mathematical object that matches this boundary is called a *surface*.

a. Bend a sheet of paper to represent the surface of a pipe.

b. Bend a sheet of paper to represent the surface of an ice-cream cone.

c. Hold a sheet of paper so that it represents the surface of your desk.

d. Which of these surfaces is flat? Which are not flat?

A flat surface, assumed to extend indefinitely so that it has no edge, is called a *plane*.

4. A straight line is a set of points. We can imagine the points of a straight line as being arranged like beads on a string. It is customary to use capital letters as names or labels for points.

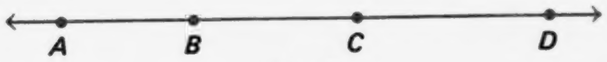


Fig. 5

In Figure 5 the points labeled A, B, C, and D are all in the same straight line.

a. Which of them are *between* A and D?

b. Which of them is between B and D?

c. Are there any unlabeled points in the line between A and B?

d. If R and S are any two different points in a straight line, are there any other points between them in the straight line?

e. How many points are there in a straight line?

f. Are there ways in which the points in a straight line are not like beads on a string?

5. A rubber band stretches easily when it is handled. A cloth napkin does not stretch easily, but it is easily bent and folded. When we handle something that stretches or bends easily, it may change its size or shape from moment to moment. On the other hand, a metal spoon, a wooden bowl, and a china plate do not stretch or bend easily under ordinary conditions. Things that do not stretch or bend are called *rigid bodies*. A rigid body keeps its size and shape when it is handled or moved. The mathematical idea that we use to match the property of rigidity of physical objects is called *congruence*. Two geometric figures are congruent if a rigid body that fits exactly over one of them will also fit exactly over the other. For example, if a dime is placed on a sheet of paper in two different positions and in each position a circle is drawn around the edge of the dime, then the two circles are congruent because the rigid edge of the dime fits both circles exactly.

a. A metal rod expands when it is heated and contracts when it is cooled. Is a metal rod absolutely rigid?

b. Under what conditions may we safely think of a metal ruler as being approximately rigid?

6. In the preceding discussion exercises we introduced three kinds of geometric objects—*points*, *straight lines*, and *planes*. We also introduced two kinds of geometric relations—*betweenness* and *congruence*. Each of these geometric objects or relations matches something we have experienced in physical space. In each case the geometric object or relation is an idealization of the physical object or relation that it matches. A geometric object or relation is only an idea. To help us think about geometric ideas, we sometimes represent them by physical objects or relations that match them. But no physical object ever represents perfectly the geometric idea that it matches, so the representation is only approximate. For example, we may represent a mathematical line by a pencil line, but a pencil line is not exactly like a mathematical line. It is only almost like a mathematical line. Why?

Exercises—2

1. Which more nearly represents a point—a pebble or a grain of sand? A dot made with a blunt pencil or a dot made with a finely sharpened pencil?

2. When physicists describe the motion of the earth around the sun, they picture the earth as a point. Why does this make sense, even though the diameter of the earth is about eight thousand miles?

3. Each of the following physical objects is approximately like a straight line: a taut string stretched between two points; a single crease made by folding a sheet of paper; the path of a ray of light; the path of a rifle bullet over a short distance. Which of these is normally used by—

- **a.** A hunter when he aims his rifle; when he fires his rifle?
- **b.** A gardener when he plants a straight row of seeds?
- **c.** A first-grade pupil who wants to divide his paper into columns?
- **d.** A bricklayer building a straight wall?

4. Grasp one end of a stiff wire between the thumb and forefinger of your left hand, and grasp the other end between the thumb and forefinger of your right hand. Twirl the wire between your fingers while keeping the ends in a fixed position.

- **a.** If the wire is curved, do all of its points stay in a fixed position?
- **b.** If the wire is straight, do all of its points stay in a fixed position?
- **c.** If a rigid body moves so that two given points in the body stay fixed, we say that the body *rotates* or *spins*. If a rigid

body moves so that two given points in the body stay fixed, what other points of the body are also fixed?

5. Which of the following objects may be used to represent approximately a portion of a plane: the surface of a hill, the surface of a pond on a calm day, the surface of an ordinary windowpane, the inside surface of a cup?

6. Which is more nearly a rigid body—a sheet of paper or a sheet of wallboard?

7. **a.** Which will give more accurate measurements—a cloth tape measure or a steel tape measure?
  b. Why?

8. **a.** Describe two ways in which you can compare the lengths of two straight sticks.
  b. How does each method depend on the properties of a rigid body?

On page 3 we listed three steps that are taken in the construction of a mathematical model of physical space. In Class Discussion 2 and in the exercises above we gave examples of the first step in the construction of Euclidean geometry. The remainder of this unit contains many examples of the second and third steps.

This unit is not intended to provide you with a complete development of Euclidean geometry. It has these more limited purposes:

(1) To introduce you to some properties of Euclidean geometry that are useful in everyday life.

(2) To give you many opportunities to make your own discoveries in geometry, either by observation or by reasoning.

(3) To give you some understanding of the nature and the uses of a mathematical model.

3 Points, Straight Lines, and Planes

Undoubtedly you are already familiar with many of the properties of points, straight lines, and planes. Class Discussion 3 and the exercises that follow provide you with an opportunity to review some of the properties of points, straight lines, and planes that are used later on in this unit.

Class Discussion 3

1. Use a flat sheet of paper to represent a plane.

  a. Place a dot on the paper to represent a point in the plane. Label this point *P*. Draw a straight line in the plane that passes through *P*. Draw a second straight line through *P*. Draw a third straight line through *P*.

  b. How many straight lines are there in the plane that pass through *P*?

  c. In any plane, how many straight lines are there that pass through a given point in the plane?

2. a. Again use a flat sheet of paper to represent a plane. Label two points in the plane *R* and *S*. Use the edge of your ruler to represent a straight line. Place the edge of the ruler so that it passes through both *R* and *S*. When the ruler is in this position, and if the paper extends beyond each end of the ruler, are there any points of its edge that do not lie on the paper?

  b. If a straight line passes through two points of a plane, how many points of the straight line are not in the plane?

3. a. Label two points in a plane *R* and *S* respectively. Draw a straight line that passes through both *R* and *S*.

  b. Is there another straight line, different from the one that you drew in exercise 3a, that also passes through both *R* and *S*?

  c. How many straight lines are there that pass through two given points?

d. If a straight line passes through two points R and S, we can name or label the line "straight line RS." For the sake of brevity the word "straight" is often omitted and we simply say "line RS." Could the line through points R and S be labeled in any other way?

4. a. Let P be a point in the plane represented by the top of your desk, and let Q be a point above the plane. Place a book on your desk so that its spine passes through P and Q. Then the spine of the book may be used to represent line PQ. If the book is open, so that pages touch only at the spine, each page represents a plane that contains line PQ.

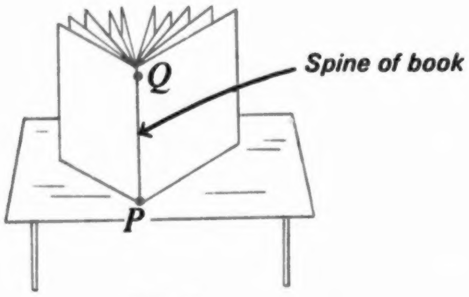


Fig. 6

b. How many planes are there that contain two given points?

c. How many planes are there that contain a given straight line?

d. Let R be a third point that is not in line PQ. Use your fingertip to represent point R. Turn a page of the book until it touches your fingertip.

e. How many planes are there that contain a given line and a given point that is not in the line?

f. How many planes are there that contain three given points that are not all in the same straight line?

5. a. Draw a straight line; choose any point in the line; and label the point P. Locate two points A and B in the line so that P is not between them. We say that A and B are on the same side of point P.

b. Locate a point C in the line so that P is between A and C. We say that A and C are on opposite sides of point P.

c. Locate a point D different from C so that A and D are on opposite sides of point P. (That is, P is between A and D.)

d. Is P between C and D?

e. Are C and D on the same side of P, or on opposite sides of P ?

f. How many sides of P are there in the line?

Any given point in a straight line divides the points of the line into three sets of points. One set consists of the given point alone. A second set consists of all points in the line that are on one side of the given point. The third set consists of all points in the line that are on the other side of the given point.

6. a. Draw a straight line PQ in a plane. Line PQ divides the points of the plane into three sets. One set consists of the points in line PQ. A second set consists of the points of the plane that are on one side of line PQ. The third set consists of the points of the plane that are on the other side of line PQ.

b. Let A and B be two points of the plane that are on the same side of line PQ. Draw line AB.

c. Is there a point in line PQ that is in line AB and is between A and B?

d. Let C and D be two points of the plane that are on opposite sides of line PQ. Draw line CD.

e. Is there a point in line PQ that is in line CD and is between C and D?

f. What is a simple test that will show if two points of the plane are on the same side or on opposite sides of line PQ?

7. Just as a point in a line divides the points of the line into three sets, and a line in a plane divides the points of the plane into three sets, so also a plane divides the points of space into three sets. Discuss these three sets.

8. Let A and B be two points in a straight line. The set of points consisting of A and B and all points between A and B is called *segment* AB. Obviously segment AB is the same as segment BA. The points A and B are called the *end-points* of segment AB. If we choose a particular segment as a *unit segment*, we can assign to every segment a measure called its *length*. The length of a segment can be found by measuring it with a ruler that is divided into segments that are congruent to the unit segment. (See exercise 5, page 6, for a discussion of congruence.) The length of a segment is also the *distance* between its endpoints.

a. If two segments have the same length, are they congruent? Why?

b. If two segments are congruent, do they have the same length?

c. Let segment *AB* be given, and let *C* be a point in line *EF*. How would you use a ruler to locate a point *D* in line *EF* so that segment *CD* is congruent to segment *AB*?

d. You can also locate the required point *D* in line *EF* by using a pair of compasses instead of a ruler. (See Figure 7.) Open the pair of compasses until the pin point rests on *A* and the pencil point rests on *B*. Then move the pair of compasses as a *rigid body*; place the pin point at *C*, and use the pencil point to draw an arc crossing line *EF*. Label as *D* the point at which the arc crosses line *EF*. Segment *CD* is congruent to segment *AB*. We also say that segment *CD* is a *copy* of segment *AB*.

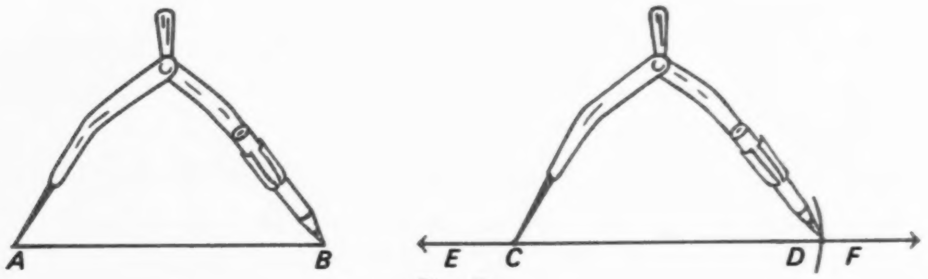


Fig. 7

e. If *A*, *B*, and *C* are three points in the same line, is the statement "the length of segment *AB* + the length of segment *BC* = the length of segment *AC*" necessarily true?

f. Under what conditions is it true?

9. The *intersection* of two sets of points consists of all points that are in both sets.
   - a. Can there be as many as two points in the intersection of two different straight lines?
   - b. What is the greatest number of points there can be in the intersection of two different straight lines?

If the intersection of two straight lines consists of one point, we say that the lines *intersect* at that point. If the intersection of straight lines contains no points—that is, if the intersection is empty—then we say that the two lines *do not intersect.* If two straight lines do not intersect and there is no plane that contains both lines, we say that the lines are *skew.* If two straight lines do not intersect and there is a plane that contains both lines, we say that the lines are *parallel.*

Two segments are said to be parallel if the straight lines that include them are parallel. The two straight lines that contain the top and bottom edges of this page are an example of parallel lines. To obtain an example of two skew lines, cross your two forefingers and then separate them as shown in Figure 8. The separated fingers represent a pair of skew lines.

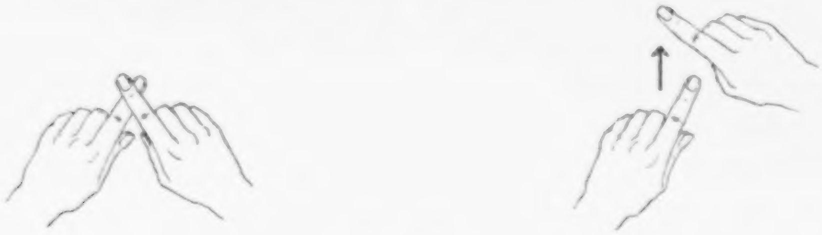


Fig. 8

In exercise 3d on page 10 it was explained that a line can be labeled by naming two of its points. Sometimes, however, it is more convenient and less confusing to use a single letter. In the next exercise it makes things clearer to use a single letter in naming a line (Figure 9).

10. When we draw a straight line, we cannot draw all of it. We can draw only part of it, and we must imagine the rest of it as an extension of the part that we drew. If we draw two straight lines in

a plane, and the parts that we have drawn have a point in common, then we can be sure that the lines intersect.

a. If the parts that we have drawn have no point in common, can we be sure that the lines do not intersect?

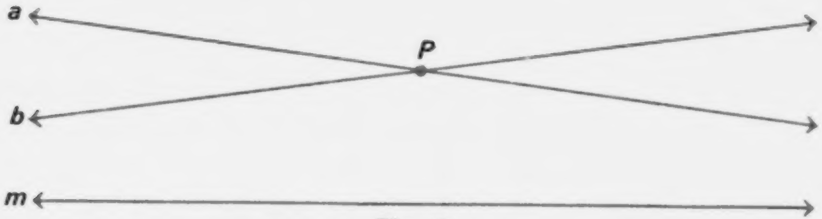


Fig. 9

b. In Figure 9, lines *a*, *b*, and *m* are straight lines in the plane of the paper. Lines *a* and *b* pass through point *P*, but *P* is not in line *m*. Note that the parts of lines *a* and *b* that are drawn have no point in common with the part of line *m* that is drawn. Does this fact guarantee that lines *a* and *m* do not intersect?

We can imagine a diagram like Figure 9 in which the parts of lines *a*, *b*, and *m* have each been drawn 100 million miles long, but in which the parts of *a* and *b* still do not intersect line *m*. Although 100 million miles is greater than any length we can really draw, we still would not be sure whether lines *a* and *m* are parallel, and we would not be sure whether lines *b* and *m* are parallel.

c. Suppose we want to find out how many lines that pass through point *P* are parallel to line *m*. Can we settle this question by drawing all lines in the plane of the paper that pass through *P* and observing how many do not intersect *m*?

No, we cannot. The answer to the question depends on the choice of a mathematical model of physical space. As explained on page 4, three such models have been constructed. Each is based on an assumption that gives us a different answer to the question.

Assumption (1): Through a point not in a given straight line there are no straight lines parallel to the given line.

Assumption (2): Through a point not in a given straight line there is exactly one straight line parallel to the given line.

Assumption (3): Through a point not in a given straight line there is more than one straight line parallel to the given line.

If Assumption (1) is made, the model of space is elliptic geometry. If Assumption (2) is made, the model of space is Euclidean geometry. If Assumption (3) is made, the model of space is hyperbolic geometry. Since we are using the Euclidean model, we make Assumption (2).

Exercises—3

1. Name the four segments that are the sides of the four-sided figure shown in Figure 10.

2. Name the three segments that are the sides of the three-sided figure shown in Figure 11.

3. a. In Figure 10, name two sides that do not have a common endpoint. These are called *opposite* sides of the figure.
   b. Name another pair of opposite sides in Figure 10.

4. In Figure 10, which sides appear to be parts of parallel lines?

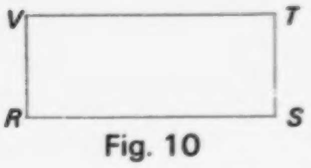


Fig. 10

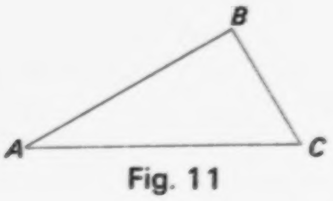


Fig. 11

5. Let points A, B, and C be three points in the same line. Under what conditions is the length of segment AC equal to the length of segment AB minus the length of segment BC?

6. Figure 12 is a diagram of the surface of a brick. The dotted lines represent edges that are not visible because they are hidden behind the brick.

a. Name a pair of edges that have a common point.

b. Name a pair of edges that are parts of parallel lines.

c. Name a pair of edges that are parts of skew lines.

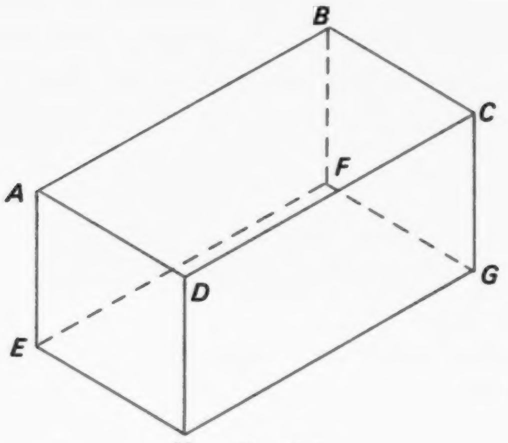
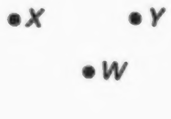


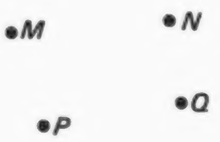
Fig. 12

7. a. The two dots at the right that are labeled *A* and *B* represent two points. How many lines are there that contain both points?

b. Think of three points not all in the same straight line. How many different lines can be drawn so that each line contains two of these points?



c. Now consider four points like those suggested by the diagram at the right. Notice that no three of the points are in the same straight line. How many lines can be drawn so that each line contains two of these points?



d. Using the same procedure as suggested above, find the number of lines determined by five points, no three of which are in the same straight line; by six points; by seven points. Record your results in a table like the one at the right.

Points	Lines
2	1
3	3
4	6
5	
6	
7	

e. Without experimenting, see if you can answer the following question. If eight points, no three of which are in the same straight line, were connected in pairs, how many segments would there be? Check your conclusion by experimenting with eight points.

f. If twelve points, no three of which are in the same straight line, were connected in pairs, how many segments would there be?

4 Circles

Let A be any point in a plane. Open a pair of compasses until the pin point and pencil point are a given distance apart. With the pin point at A, draw a curve with the pencil point. The curve that you get is called a *circle*. The point A is called the *center* of the circle. The given distance is called the *radius* of the circle. It is clear from the way in which the circle is drawn that the distance from each point in the circle to the center of the circle is equal to the given distance.

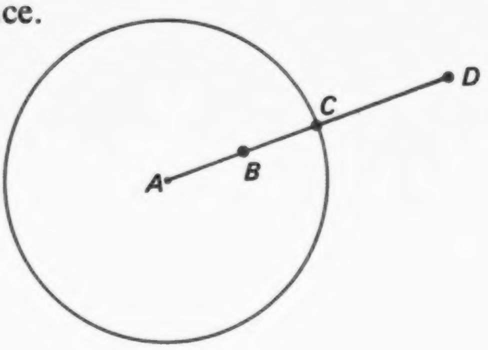


Fig. 13

Class Discussion 4

1. In Figure 13, point A is the center of the circle, and point C is in the circle.
   - **a.** Which distance, the distance between A and B, A and C, or A and D, is equal to the radius of the circle?
   - **b.** Which of these distances is less than the radius?
   - **c.** Which of these distances is greater than the radius?

If the distance of a point from the center of a circle is less than the radius, the point is said to be in the *interior* of the circle. If the distance of a point from the center of a circle is greater than the radius, the point is said to be in the *exterior* of the circle.

- **d.** In Figure 13, which point, B or D, is in the interior of the circle?
- **e.** Which point is in the exterior of the circle?

A circle divides the points of a plane into three sets: one set consists of the points in the circle; a second set consists of the points in the interior of the circle; and the third set consists of the points in the exterior of the circle.

2. **a.** Draw a circle. Choose any two points B and C in the interior of the circle. Draw the segment that joins these two points.
   - **b.** Does segment BC cross the circle?
   - **c.** Does segment BC lie entirely in the interior of the circle?
   - **d.** Choose any two points D and E in the exterior of the circle. Draw the segment that joins them.
   - **e.** Are there cases where this segment does not cross the circle?
   - **f.** Are there cases where this segment does cross the circle?
   - **g.** Draw a connected line, not necessarily straight, from D to E without crossing the circle. (A connected line is one without any breaks in it.)
   - **h.** Can this always be done?
   - **i.** Draw a connected line from B to D.
   - **j.** Does it cross the circle?
   - **k.** Is it possible to draw a connected line from B to D that does not cross the circle?

l. Can any two points in the interior of a circle be joined by a segment that does not cross the circle?

m. Can any two points in the interior of a circle be joined by a connected line that does not cross the circle?

n. Can any two points in the exterior of a circle be joined by a segment that does not cross the circle?

o. Can any two points in the exterior of a circle be joined by a connected line that does not cross the circle?

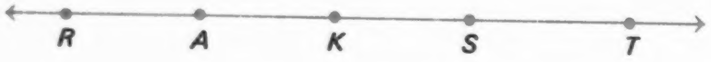
p. Can a point in the interior of a circle be joined to a point in the exterior of the circle by a connected line that does not cross the circle?

3. A set of points is said to be *convex* if for every pair of points in the set, the segment that joins them lies entirely in the set.

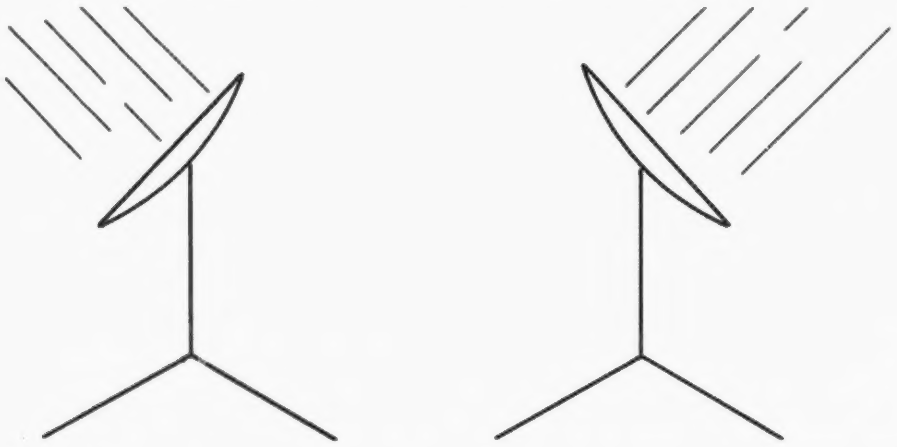
a. Is the interior of a circle convex?
b. Is the exterior of a circle convex?

5 Angles

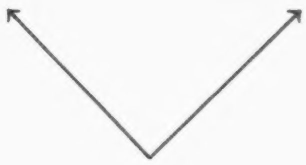
Let A be a point in a line. The set of points consisting of A and all points in the line that are on one side of A is called a *ray*. The point A is called the endpoint of the ray. To label a ray, we use a pair of letters. The first letter in the pair indicates the endpoint of the ray, and the second letter indicates one other point of the ray. Thus in the figure below the ray consisting of point A and all points to the right of A is called "ray AK," or "ray AS," etc. How many different rays included in the same line can have the same endpoint? Two different rays in the same line that have the same endpoint are called *opposite* rays. For example, ray AK and ray AR are opposite rays.



To attract attention to the opening of his used-car lot, a dealer rented two searchlights. The lights scanned the sky with intense rays of light that could be seen for miles.

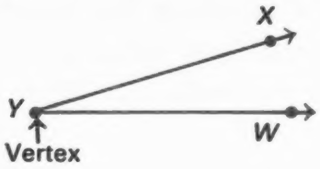


To an observer several blocks away, the searchlights probably looked something like the figure shown at the right. In geometry, a figure like this is called an *angle*.



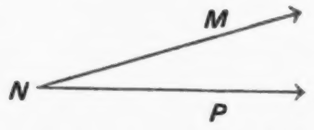
An angle is a set of points consisting of two rays that have a common endpoint but are not in the same line. The common endpoint of the rays is called the *vertex* of the angle. The rays are called the *sides* of the angle.

An angle may be named in a variety of ways. One way is to make use of three letters. If this scheme is used, the letter that labels the vertex is written between the other two. Accordingly, the angle pictured at the right can be named either "angle *XYW*" or "angle *WYX*." Symbolically we can write these names as $\angle XYW$ and $\angle WYX$. If only one angle in a diagram has a given vertex, we can name the angle with the letter at the vertex. Thus the angle pictured above might also be called "angle *Y*."

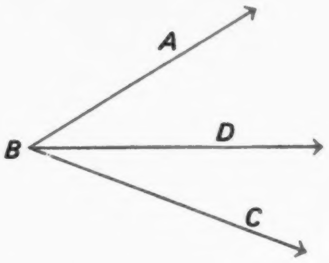


Class Discussion 5a

1. Name the angle pictured at the right in three different ways.



2. Name all angles represented in the drawing below. (Note that the name "angle *B*" is not clear enough to identify a particular angle.)

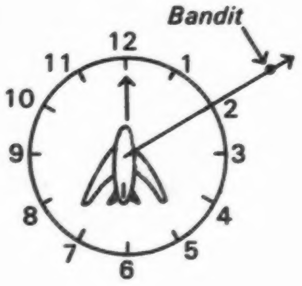


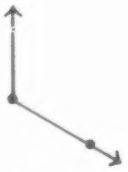
3. The figure formed by the hands of a clock may be thought of as representing an angle. Make drawings of the angles suggested by the following clock readings:

a. 3:00 **b.** 5:30 **c.** 4:10 **d.** 8:55 **e.** 2:50

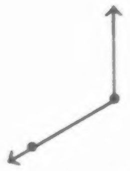
Exercises—5a

1. An aircraft pilot often calls another aircraft in flight a bandit. The pilot refers to the position of the bandit with respect to his plane as being at 2 o'clock, 3 o'clock, etc. When this method is used, the pilot assumes that he is flying at the center of a clock and is heading toward 12.





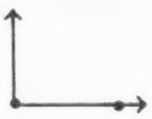
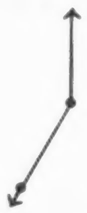
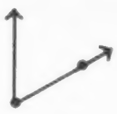
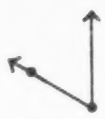
Bandit at 4 o'clock



Bandit at 8 o'clock

a. Make a sketch of the following situations.
  (1) Bandit at 2 o'clock
  (2) Bandit at 6 o'clock
  (3) Bandit at 9 o'clock
  (4) Bandit at 11 o'clock

b. Write the pilot's description for each of the following sketches.

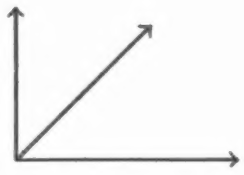





2. Let's see how good you are at counting angles.

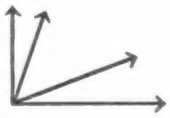
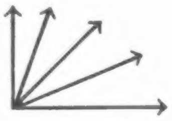
a. How many angles are formed by two rays that have a common endpoint if the two rays are not in the same line? Record your result in a table like the one at the right.

Number of Rays	Number of Angles Formed
2	
3	
4	
5	

b. How many angles are formed by three rays that have a common endpoint if no two of the rays are in the same line? Record this information in your table.



c. How many angles are represented in each of the figures shown at the right?

d. Without actually counting, make a prediction regarding the number of angles formed by six rays that have a common endpoint if no two of the rays are in the same line.

e. Test your prediction in exercise 2d by drawing an appropriate figure and counting the angles.

f. How many angles are formed by nine rays that have a common endpoint if no two of the rays are in the same line?

3. Consider the formulas below. Which one can be used to find the number of angles formed by a given number of rays that have a common endpoint if no two of the rays are in the same line? In each formula, A represents the number of angles and R represents the number of rays.

$$A = R - 1 \qquad A = 2R - 3 \qquad A = R^2 - 3 \qquad A = \frac{R \cdot (R - 1)}{2}$$

Class Discussion **5b**

An angle divides the points of a plane into three sets. One set consists of the points of the angle; a second set consists of the interior of the angle; and a third set consists of the exterior of the angle. In Figure 14, points D and E are in the interior of angle ABC. Points F and G are in the exterior of angle ABC.

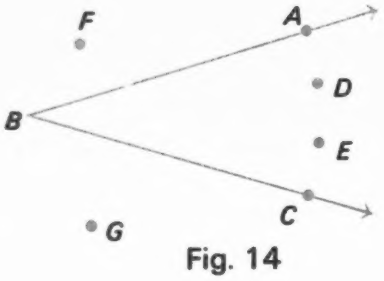


Fig. 14

1. a. Can D and E be joined by a segment that does not cross the angle?

b. Can F and G be joined by a segment that does not cross the angle?

c. Can F and G be joined by a connected line, not necessarily straight, that does not cross the angle?

d. Can *F* and *D* be joined by a connected line that does not cross the angle?

e. Can any two points in the interior of an angle be joined by a connected line that does not cross the angle?

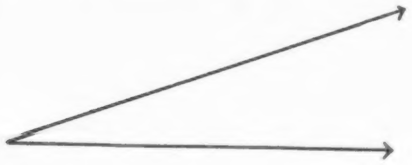
f. Can any two points in the exterior of an angle be joined by a connected line that does not cross the angle?

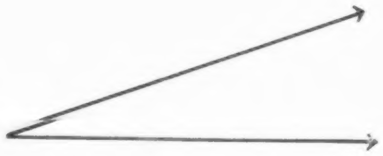
g. Can a point in the interior of an angle be joined to a point in the exterior of the angle by a connected line that does not cross the angle?

h. Is the interior of an angle convex? (See exercise 3, page 19.)

i. Is the exterior of an angle convex?

2. a. Make a tracing of the angle on the left. Compare the two angles by placing the tracing of the angle on the left over the angle on the right.





b. Are the two angles congruent? (See page 6 for a discussion of congruence.)

3. In Figure 15 below, lines *AB* and *CD* intersect at *E*. Note that the letters *w*, *x*, *y*, and *z* have been placed in the interiors of the angles formed by the two lines. These letters may be used as names for the angles. Thus "angle *w*" means "angle *DEB*."

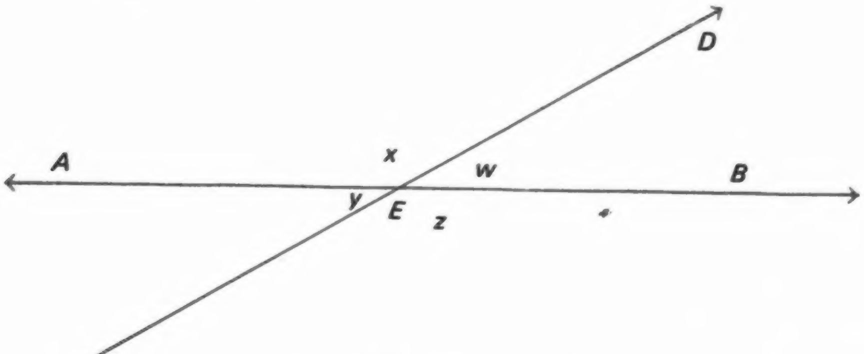


Fig. 15

a. Make a tracing of the above diagram on a sheet of paper and use the tracing to compare the various angles.
b. Which angle is congruent to angle *w*?
c. Which angle is congruent to angle *x*?
d. Is angle *w* congruent to angle *x*?

4. Figure 16 below contains only that part of Figure 15 that pictures angles *w* and *x*.

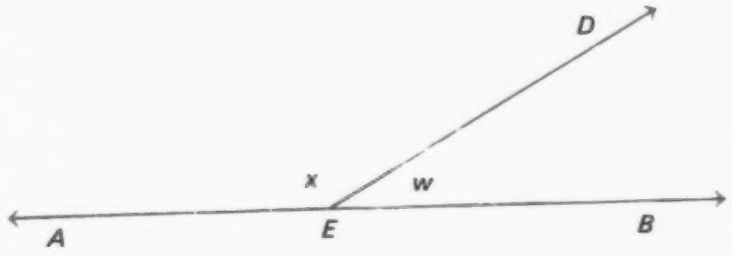


Fig. 16

Angles *w* and *x* have these properties:

(1) They have the same vertex.
(2) They have a common side.
(3) The other two sides are opposite rays.

Two angles that have properties (1), (2), and (3) are called a *linear pair* of angles. Thus angles *w* and *x* are a linear pair. Name three other linear pairs of angles in Figure 15.

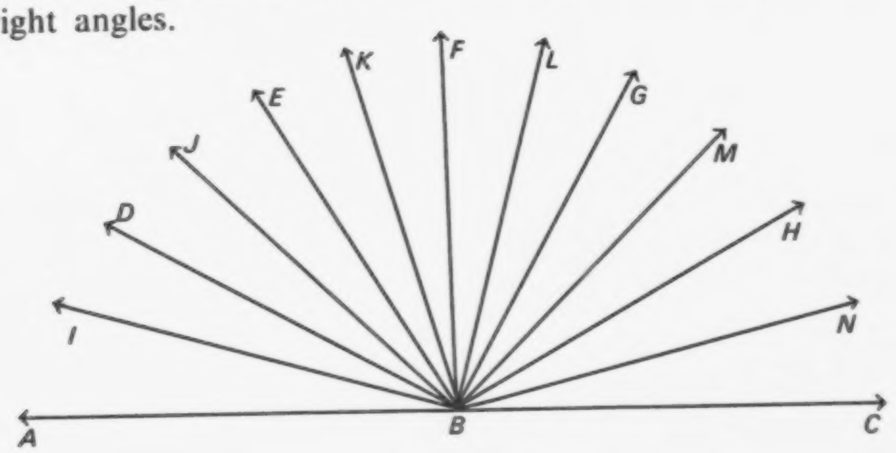
5. If the two angles of a linear pair are congruent, then each angle of the linear pair is a *right angle*. Make a drawing of a linear pair of angles that are congruent.

6. If two lines intersect so that right angles are formed, then the two lines are said to be *perpendicular*. Make a drawing that shows two lines that are perpendicular. How many right angles are formed by two perpendicular lines?

Exercises—5b

1. Find some examples of objects in your classroom that represent right angles.

2. In the diagram below name all angles that appear to be right angles.



6 Measure of an Angle

Class Discussion 6a

If a right angle is divided into 90 congruent angles, each of them is called a *degree*. The degree is the unit angle for a system of measures of angles. The symbol for a degree is °. Because a right angle contains 90 degrees, we say that its measure is 90°. Pictured below is an angle whose measure is 90°.

1. Take a sheet of paper that has a straight edge, and fold it so that the straight edge falls along itself. Press the crease flat, then open the paper again. The crease and the edge form two right angles. The intersection of the crease and the edge of the paper is the vertex of both right angles. Cut the sheet of paper along the crease into two pieces.
   a. Take one of the pieces and fold it so that a right angle is divided into two congruent angles. How many degrees are there in each of these angles?

b. Fold the other piece twice so that a right angle is divided into three congruent angles. How many degrees are there in each of these angles?

c. On the basis of your results in exercises 1a and 1b can you conclude that *congruent angles have the same measure*, and that *angles that have the same measure are congruent?*

2. By folding paper as in exercise 1, form an angle whose measure is 15°; 30°; 45°; 60°; 75°; 90°; 105°; 135°; 150°; 165°.

3. **a.** Which of the angles pictured in Figure 17 have measures that are less than 90°?

b. Which of the angles have measures that are greater than 90°?

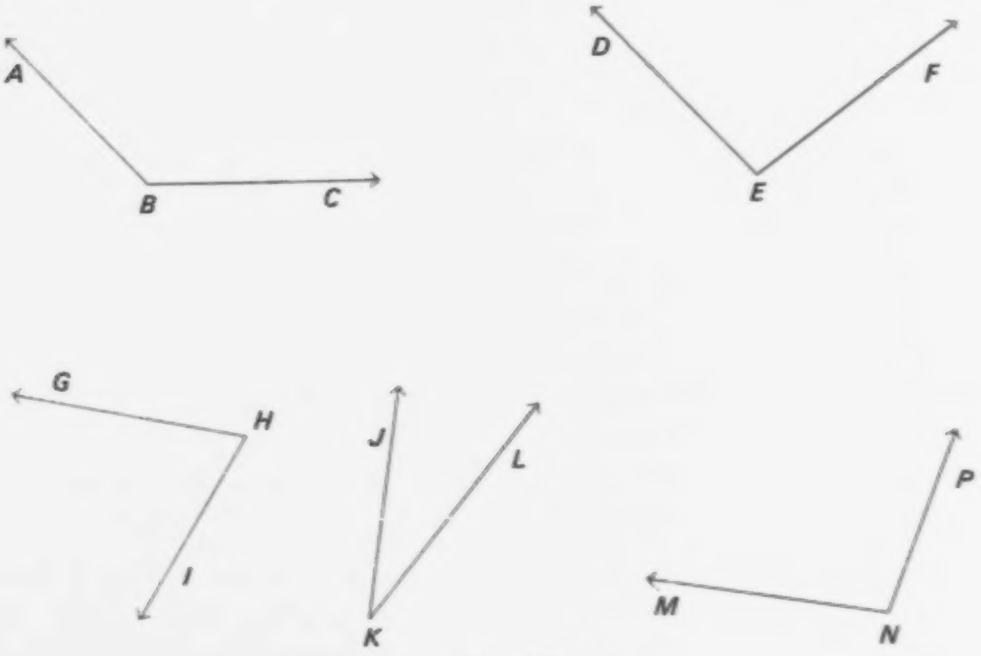


Fig. 17

4. **a.** Sketch an angle that has a measure of approximately 45°.

b. Sketch an angle that has a measure of approximately 30°.

c. Sketch an angle that has a measure of approximately 15°.

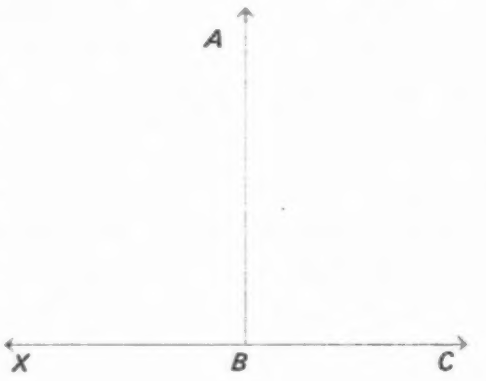
d. Sketch an angle that has a measure of approximately 60°.

e. Sketch an angle that has a measure of approximately 75°.

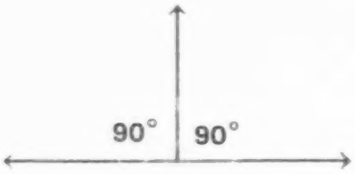
5. In the last exercise you sketched various angles. Now you are going to sketch these angles again. However, this time you will put them all on one diagram. Follow the directions carefully.

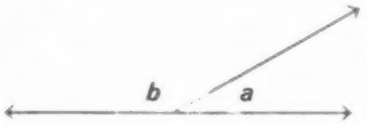
a. Make a copy of the diagram at the right and label it as indicated. Assume that angle *ABC* and angle *ABX* are right angles.

b. Sketch the angles that have the measures given in exercise 4. For each angle, let *B* be the vertex, and let ray *BC* be one of the sides.



6. How many degrees are there in the sum of the measures of two right angles?





7. How many degrees are there in the sum of the measures of two angles that are a linear pair?

8. Let *a* and *b* be a linear pair of angles. What is the measure of angle *b* if the measure of angle *a* is 40°; 60°; 90°; 75°; 72.5°?

We measure the length of a line segment by using a ruler that is divided into unit segments. In the same way, we measure an angle by using a *protractor* that is divided into degrees. A picture of a protractor is shown in Figure 18. The point *Q* is called the center of the protractor. The line *HA* is called the straightedge of the protractor. This protractor has a scale along its curved edge, with 0 at the right-hand side and 180 at the left-hand side. The 0 and the 180 are not printed on the protractor but are understood to be on the line *HA*.

To measure an angle, place the protractor over the angle, or the angle over the protractor, so that the center of the protractor is at the vertex of the angle and one side of the angle passes through the 0 on the scale of the protractor. Then the other side of the angle points out the measure of the angle on the scale of the protractor. For example, in Figure 18, angle *AQB* has its vertex *Q* at the center of the protractor, and its side *QA* passes through 0 on the scale of the protractor. The side *QB* points to 25 on the scale. Therefore the measure of angle *AQB* is 25°.

Exercises—6a

1. In Figure 18, find the measure of each of the following angles.

a. Angle *AQB*
b. Angle *AQC*
c. Angle *AQD*
d. Angle *AQE*
e. Angle *AQF*
f. Angle *AQG*
g. Angle *HQG*
h. Angle *HQF*
i. Angle *HQD*
j. Angle *EQF*
k. Angle *GQF*
l. Angle *BQC*
m. Angle *FQD*
n. Angle *GQB*

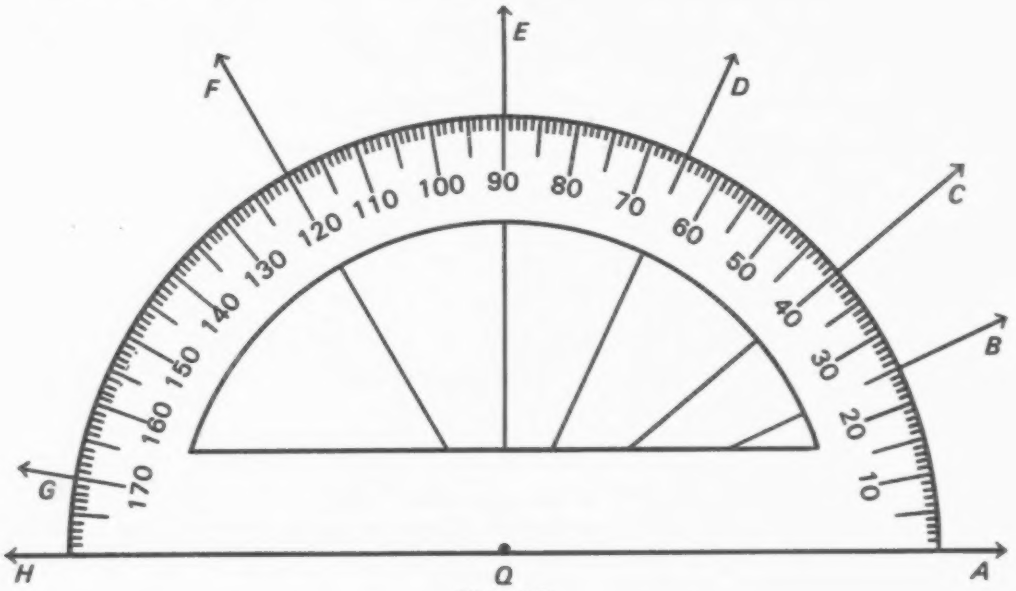
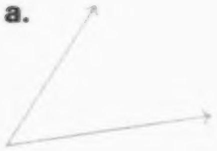
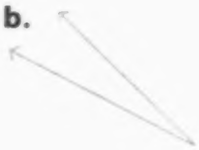
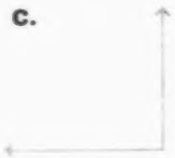


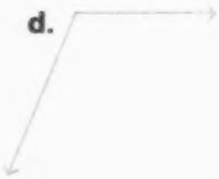
Fig. 18

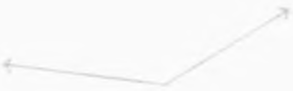
2. An *acute angle* is an angle whose measure is greater than 0° and less than 90°.
An *obtuse angle* is an angle whose measure is greater than 90° and less than 180°.
For each angle pictured below, decide whether the given angle is an acute angle, an obtuse angle, or a right angle.

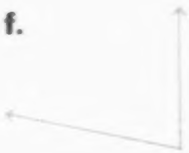
a.


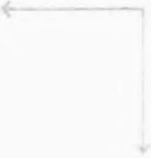
b.


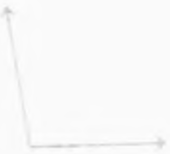
c.


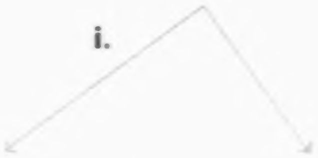
d.


e.


f.


g.


h.


i.


3. a. Which of the angle measures listed below are measures of acute angles?

35° 110° 155° 85°

b. Draw four angles that have approximately the measures listed in exercise 3a. Do not use a protractor.

c. Check the measures of the angles that you drew by placing the drawing of each over the protractor shown in Figure 18. In each case determine how much your drawing is in error.

4. Place a sheet of paper over the protractor shown in Figure 18 and draw angles that have the following measures.

a. 20°	**c.** 70°	**e.** 115°	**g.** 145°
b. 50°	**d.** 90°	**f.** 125°	**h.** 175°

5. **a.** Draw two angles whose measures have a sum of 90°.
b. Draw three angles whose measures have a sum of 180°.
c. Draw four angles whose measures have a sum of 360°.

6. What is the measure of the angle indicated by the hands of a clock when the time is—

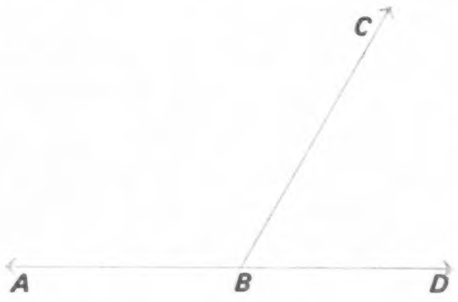
a. 3:00
b. 9:00
c. 1:00
d. 5:00
e. 11:00
f. 8:00
g. 5:30
h. 1:20

Class Discussion **6b**

1. Consider angle *ABC* and angle *DBC* shown at the right. Angle *DBC* has a measure of 60°.

a. Determine the measure of angle *ABC* without using a protractor.

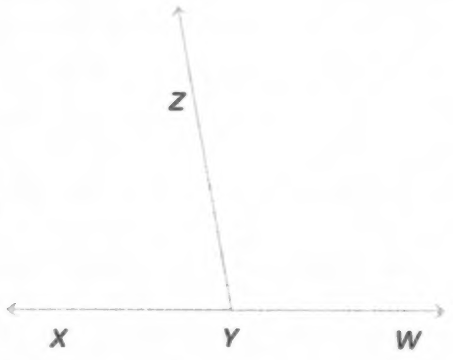
b. Use a protractor to check your answer.



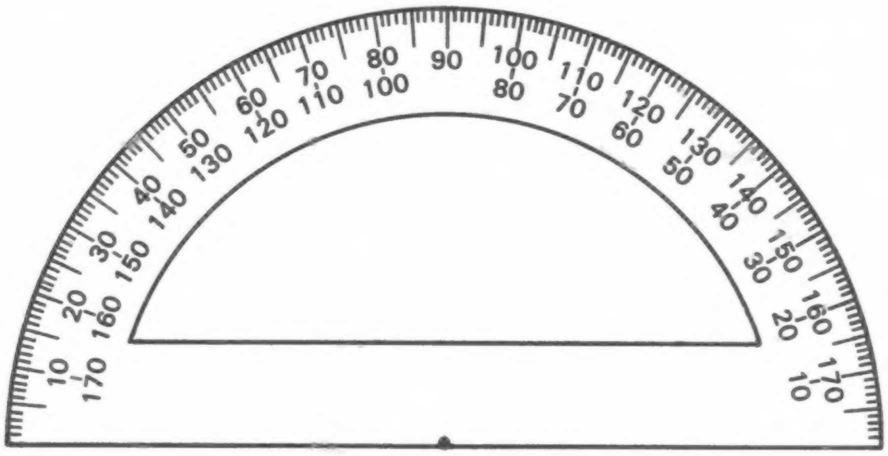
2. Consider angle *XYZ* and angle *ZYW*. Angle *XYZ* has a measure of 80°.

a. Determine the measure of angle *ZYW* without using a protractor.

b. Use a protractor to check your answer.



3. Shown below is a protractor that has two scales. Notice that there are two sets of numbers around the curved edge. One set belongs to the scale that has zero on the right. The other set belongs to the scale that has zero on the left. Each point on the curved edge of the protractor has two numbers attached to it, one from each set.



a. Examine the protractor with two scales and complete the table at the right.

b. Complete the following statements without referring to the protractor shown above.

If the inside number is 10, the outside number is____.

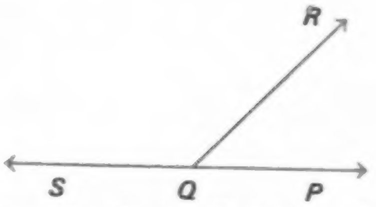
If the outside number is 70, the inside number is____.

Inside Number	Outside Number
60	
	80
40	
130	
	150
20	
	50

4. When you measure an angle with a protractor that has two scales, place the protractor so that its center is at the vertex of the angle and its straightedge is on one side of the angle. Then use the inside number if the zero of the inside scale is on one side of the angle you are measuring. If the zero of the outside scale is on one side of the angle you are measuring, use the outside number. You can also decide which number to use in another way:

a. If the angle you are measuring is acute, should you use the greater number or the lesser number?

b. If the angle you are measuring is obtuse, should you use the greater number or the lesser number?



5. **a.** In the figure at the right, is angle PQR obtuse or acute?

b. Is angle SQR an obtuse angle or an acute angle?

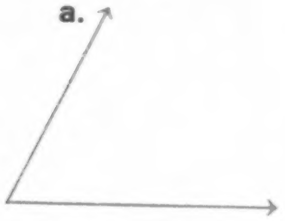
c. What should be the sum of the measures of angle PQR and angle SQR?

d. Use a protractor to find the measure of angle PQR.

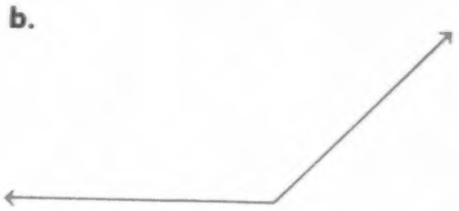
e. Use a protractor to find the measure of angle SQR.

6. In this exercise, do *not* use a protractor. Estimate the measure of each of the angles pictured below. A good way to do this is to think how each angle compares in measure with a right angle.

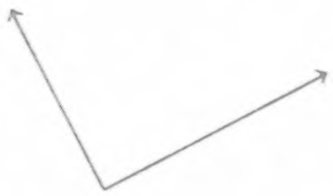
a.



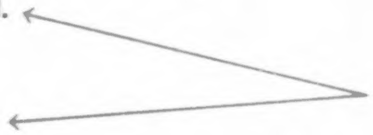
b.



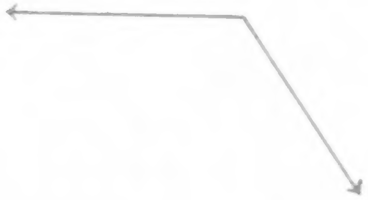
c.



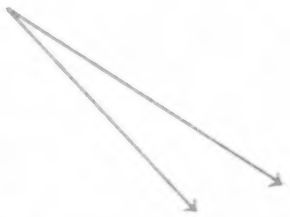
d.



e.



f.



7. Now measure each of these angles. How close were your estimates?

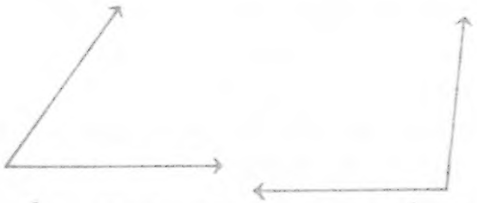
8. Draw angles which have the following measures.

a. 27°

b. 113°

Exercises—6b

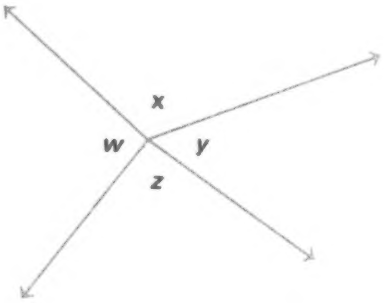
1. a. Use a protractor to find the measure of each angle shown at the right, and record your results.



b. Draw angles that have the same measures as those you recorded above.

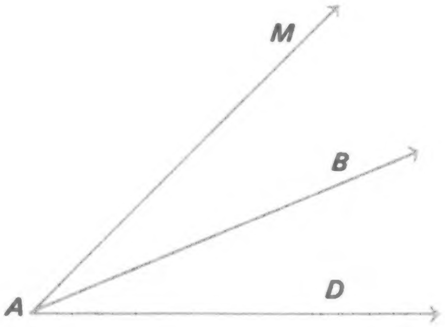
c. Now place your drawings over the angles shown in exercise 1a. How do they compare? (If you have an error that is more than 5 degrees, you should repeat the steps in exercises 1a and 1b.)

2. a. Find the measures of angles *w*, *x*, *y*, and *z*, and record your results.



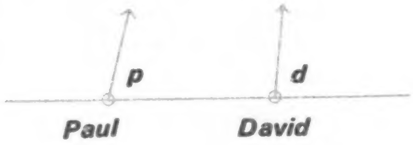
b. Can you suggest (without remeasuring) a method for checking your results in exercise 2a? Explain.

3. Find the measures of angles *MAB* and *BAD*. Are the measures of the two angles the same? Is angle *MAB* congruent to angle *BAD*? (See exercise 1c, page 27.) Does ray *AB* divide angle *MAD* into two congruent angles? A ray that divides an angle into two congruent angles is called the *bisector* of the angle.



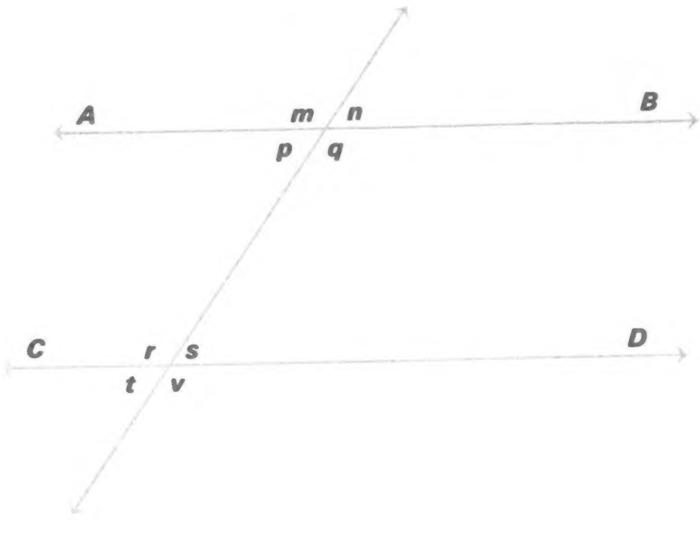
4. Paul and David are searchlight operators at an amusement park. One night they decided to try to keep their searchlight beams parallel. In the drawing below assume that the two rays extending upward represent searchlight beams.

a. If the measure of angle p is 67°, what should be the measure of angle d if the two beams are to be parallel?



b. Suppose that the measure of angle p is 75°. Make a drawing similar to the one above in which the rays that represent searchlight beams are included in parallel lines.

5. Consider the diagram below.



a. Name a pair of angles whose measures can be used to tell us whether or not lines AB and CD are parallel.

b. Name other pairs of angles whose measures can be used to decide whether or not lines AB and CD are parallel.

c. Are lines AB and CD parallel?

7 Polygons

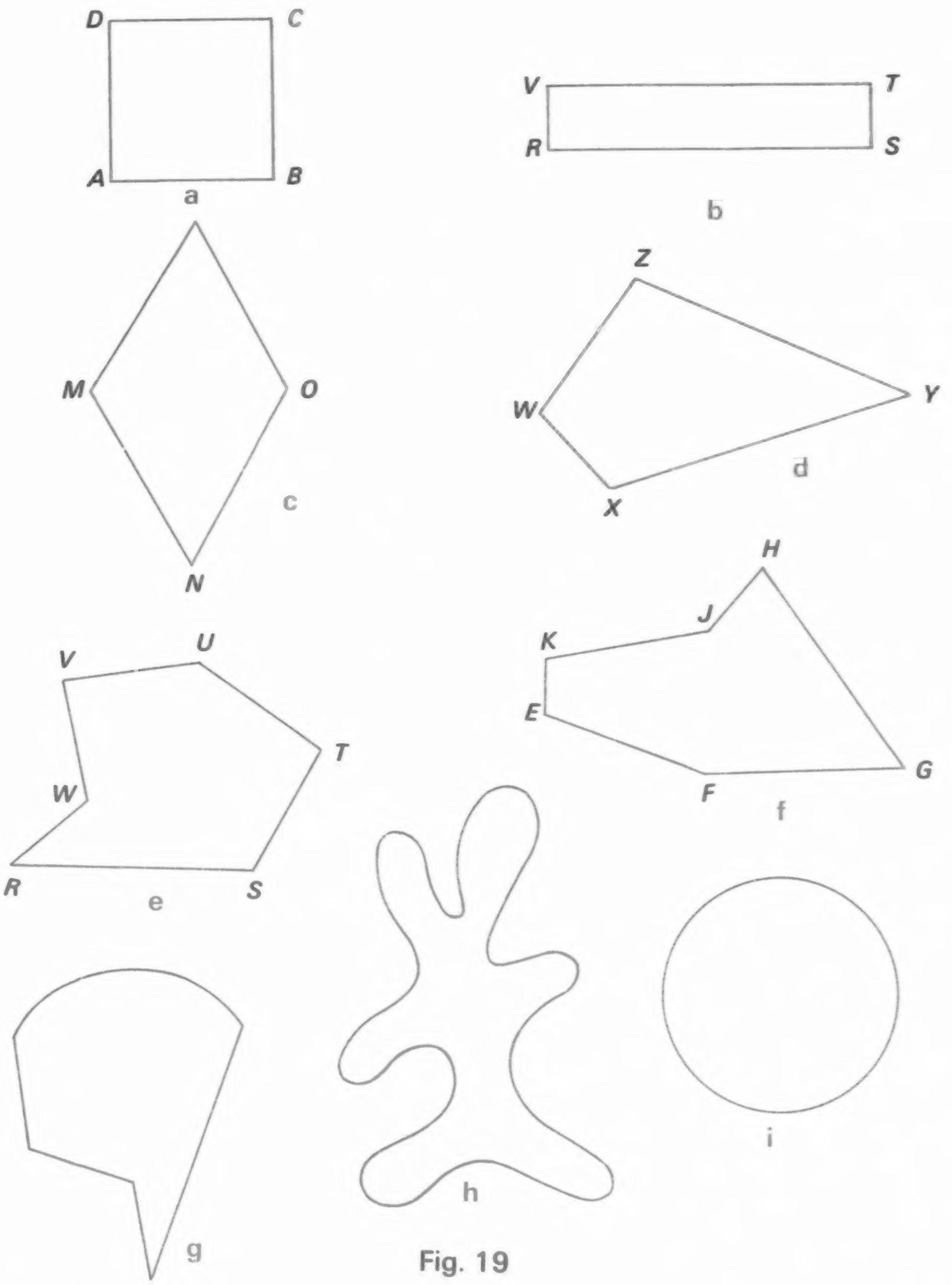

D
C
A
B
a
V
T
R
S
b
M
O
N
c
Z
W
Y
X
d
V
U
T
W
R
S
e
H
J
K
E
F
G
f
g
h
i

Fig. 19

Figures 19a through 19f represent *polygons*. Figures 19g, 19h, and 19i do *not* represent polygons. Describe some properties that you think polygons have in common.

A *polygon* is a set of points called *vertices* together with all points of certain segments called *sides* having these properties:

(1) The vertices can be numbered using counting numbers, so that there is a first vertex and a last vertex.

(2) Each vertex except the last is joined to the next vertex after it by a side; and the last vertex is joined to the first vertex by a side.

(3) Two sides intersect only at their endpoints.

(4) No two sides with a common endpoint are in the same straight line.

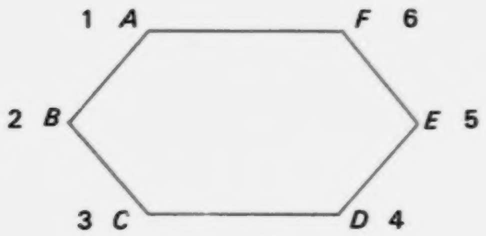


Fig. 20

Figure 20 represents a polygon. Points A, B, C, D, E, and F are vertices of the polygon. Note that the vertices are numbered from 1 to 6. Segments AB, BC, CD, DE, EF, and FA are the sides of the polygon. A polygon can be named by listing its vertices in order. Thus we can refer to the polygon above as "polygon $ABCDEF$."

Exercises—7

1. Name the sides of the polygon in Figure 19c.

2. Name the sides of the polygon in Figure 19e.

3. What is the least number of sides that a polygon can have?

4

4. How many sides does a polygon have if it has three vertices; four vertices; five vertices; any number of vertices?

5. A polygon that has three sides is called a *triangle*. A polygon that has four sides is called a *quadrilateral*. Which of Figures 19a through 19f represent quadrilaterals?

A polygon in a plane divides the points of the plane into three sets—the set that consists of the points of the polygon, a *region* called the *interior* of the polygon, and a *region* called the *exterior* of the polygon.

In Figure 21, Region I is the interior of polygon *ABCD*. Region II is the exterior of polygon *ABCD*.

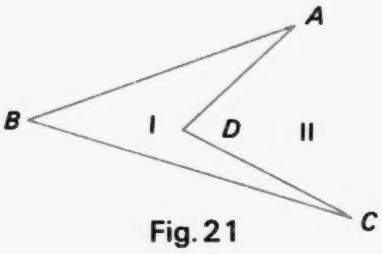


Fig. 21

6. Copy Figure 21 on your paper. Draw a circle that has all of Region I in its interior.

7. Can you draw a circle that has all of Region II in its interior?

8. What test is suggested by exercises 6 and 7 for distinguishing the interior of a polygon from its exterior?

9. A polygon is called *convex* if its interior is convex. Which of the polygons in Figures 19a through 19f are convex?

10. Can every set of two points in the interior of a polygon be joined by a segment that does not cross the polygon?

11. Can every set of two points in the interior of a polygon be joined by a connected line that does not cross the polygon?

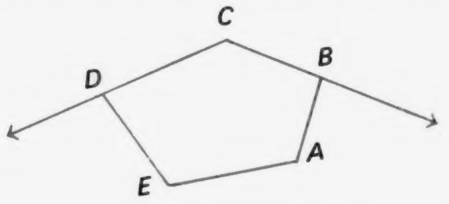
12. Can every set of two points in the interior of a convex polygon be joined by a segment that does not cross the polygon?

13. Can every set of two points in the exterior of a polygon be joined by a segment that does not cross the polygon?

14. Is the exterior of a polygon convex?

15. Can every set of two points in the exterior of a polygon be joined by a connected line that does not cross the polygon?

16. In a convex polygon we can talk about angles of the polygon. For example, in the convex polygon shown below, we say that angle *BCD* is an *angle of polygon ABCDE*. What this means is that each side of angle *BCD* includes a side of polygon *ABCDE*, and the vertex of angle *BCD* is a vertex of polygon *ABCDE*.



What is the vertex of angle *BCD*? Is the vertex of an angle of a polygon also a vertex of the polygon? Name all the angles of polygon *ABCDE*.

17. How many angles does a quadrilateral have?

18. In a quadrilateral, two sides that do not have a common endpoint are called *opposite sides*. Name the pairs of opposite sides of the quadrilaterals pictured in Figures 19a through 19d.

19. *A parallelogram* is a quadrilateral whose opposite sides are parallel. Two segments (sides in this case) are parallel if they are included in parallel lines. Which of the quadrilaterals pictured in Figures 19a through 19d are parallelograms?

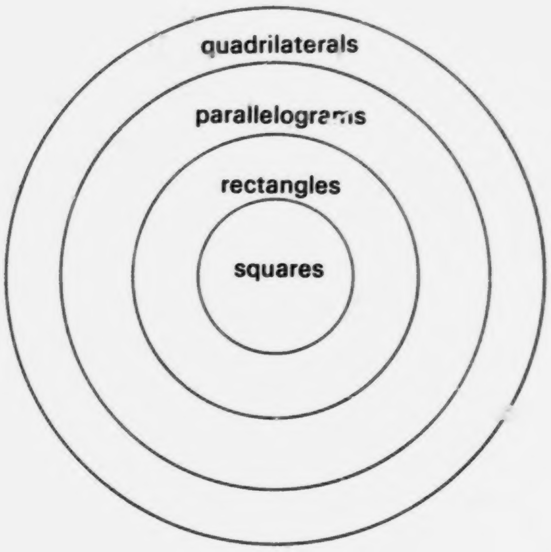
20. A *rectangle* is a parallelogram whose angles are all right angles. Which of Figures 19a through 19d represent rectangles?

21. A *square* is a rectangle whose sides are all congruent. Which of Figures 19a through 19d represent squares?

22. Is it correct to say that all squares are parallelograms? Is the converse of this statement true? (That is, are all parallelograms squares?)

23. Is it correct to say that all parallelograms are rectangles? Is the converse of this statement true?

24. Mathematicians often use diagrams to help them classify information. For example, the diagram below may be used to classify certain quadrilaterals.



a. Discuss how to use the diagram to explain the statement, "All parallelograms are quadrilaterals, but not all quadrilaterals are parallelograms."

b. Using the diagram, write similar true statements about quadrilaterals.

25. Draw a quadrilateral and label its vertices *A*, *B*, *C*, and *D*. Choose points *E*, *F*, *G*, and *H* in the sides of the quadrilateral in such a way that segments *EF* and *GH* intersect in the interior of the quadrilateral and lines *EG* and *FH* intersect in the exterior of the quadrilateral.

26. Which of the figures shown below have a pair of parallel sides?

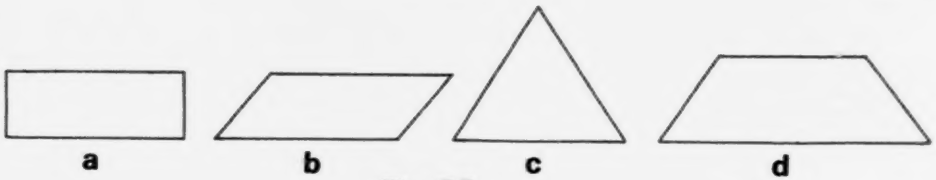
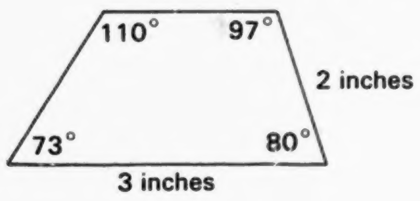


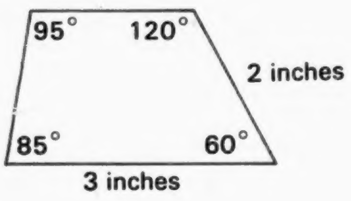
Fig. 22

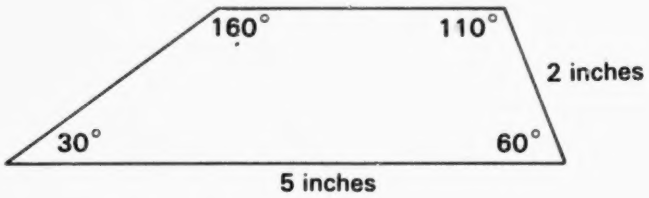
27. Does Figure 22b represent a parallelogram?

28. Does Figure 22d represent a parallelogram?

29. A *trapezoid* is a quadrilateral with two and only two sides parallel. Three quadrilaterals are sketched below. The measures of the sides and angles that are recorded in the three figures are not completely accurate for any one of the figures. Make new drawings, using the measures that are recorded. Do any of your drawings represent trapezoids?


110°
97°
2 inches
73°
80°
3 inches


95°
120°
2 inches
85°
60°
3 inches


160°
110°
2 inches
30°
60°
5 inches

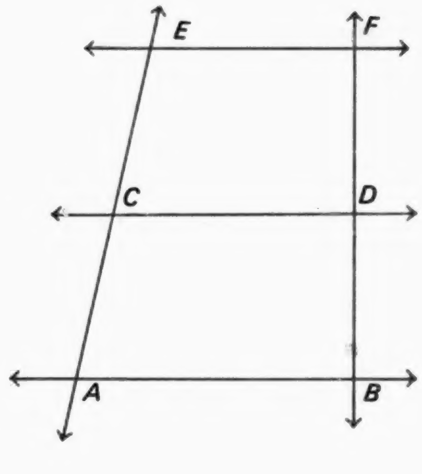
30. **a.** Name as many segments in the figure at the right as you can.

b. Are there any segments that appear to have the same length? If so, name them.

c. Which lines appear to be parallel?

d. Which lines appear to be perpendicular?

e. Name three quadrilaterals in the figure.

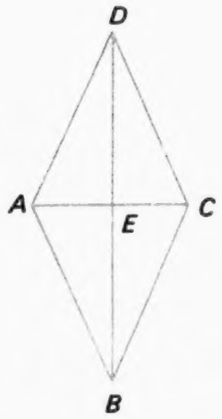


31. Consider the quadrilateral pictured at the right.

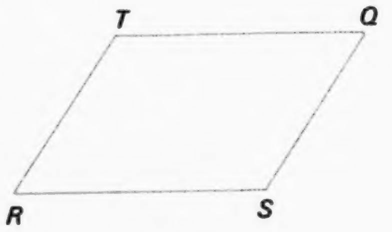
a. Name two pairs of sides that appear to be parallel.

b. If both pairs of opposite sides of quadrilateral *ABCD* are parallel, is the quadrilateral a parallelogram?

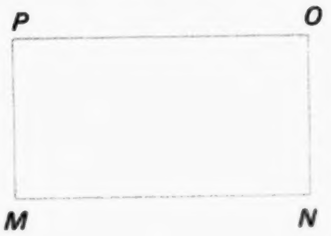
c. Name all segments that appear to have the same length.



32. a. Quadrilateral *RSQT* is a parallelogram but not a rectangle. For a parallelogram to be a rectangle, what property must it have that *RSQT* does not have?



b. Quadrilateral *MNOP* is a rectangle but not a square. For a rectangle to be a square, what property must it have that *MNOP* does not have?



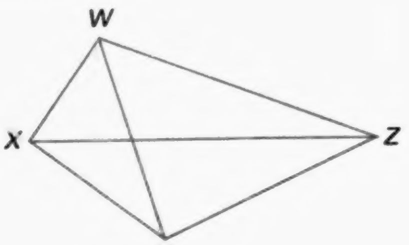
33. a. Draw a quadrilateral *ABCD* so that side *AB* is parallel to side *DC* and the angle whose vertex is *B* is a right angle. Does the quadrilateral have to be a rectangle?

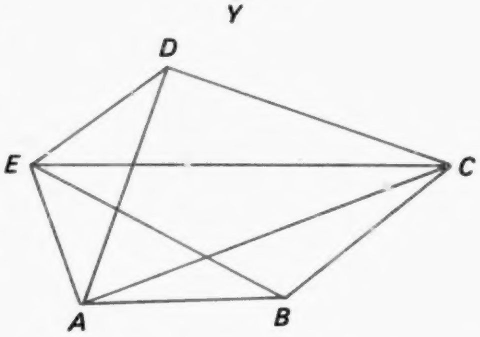
b. Draw a parallelogram *EFGH* in which all sides have the same length. Does the parallelogram have to be a square?

c. Draw a parallelogram *RSTV* in which all sides have the same length and all angles are right angles. What is the name of the figure that you are asked to draw?

8 The Diagonals of a Polygon

Class Discussion 8

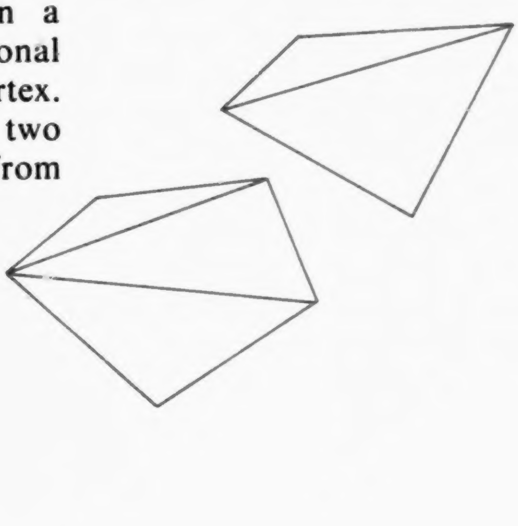




1. A *diagonal* of a polygon is a segment that is not a side of the polygon but has two vertices of the polygon as endpoints. Every quadrilateral has exactly two diagonals. In quadrilateral $WXYZ$ the diagonals are segments XZ and WY. All diagonals, except one, have been drawn in the figure at the right. Name the diagonal that has not been drawn.

2. Draw a quadrilateral in which the diagonals have the same length.

3. In this exercise you are asked to determine the number of diagonals that can be drawn from one vertex in certain polygons. For example, in a quadrilateral only one diagonal can be drawn from each vertex. In a five-sided polygon two diagonals can be drawn from each vertex.



a. Complete the chart at the right.

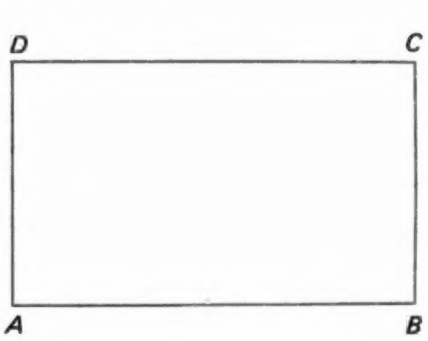
b. How many diagonals can be drawn from each vertex in a 50-sided polygon?

c. Let S represent the number of sides of a polygon. Let D represent the number of diagonals that can be drawn from one vertex. Write a formula relating D to S.

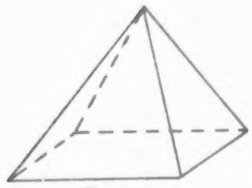
Number of Sides of a Polygon	Number of Diagonals from One Vertex
3	
4	
5	
6	
7	
8	
9	
10	

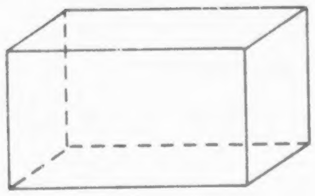
9 Polyhedrons

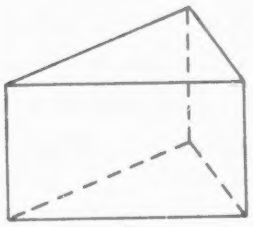
How good is your imagination? Let's test it!

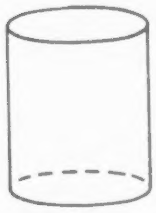


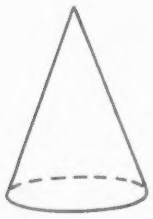
Consider rectangle $ABCD$ pictured at the right. Now think of another rectangle that is congruent to rectangle $ABCD$. Imagine that this second rectangle is located in the air above the plane of the page in such a way that each vertex is two inches above the corresponding vertex of rectangle $ABCD$. Mentally join with a segment each vertex of the second rectangle to the corresponding vertex of the rectangle directly below it. Which one of the illustrations below gives the best picture of the geometric figure that has been described? (The dotted lines in each fiqure are "hidden lines" that are behind one or more surfaces shown in the figure.)

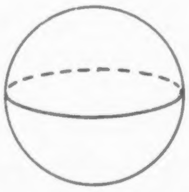

a


b


c

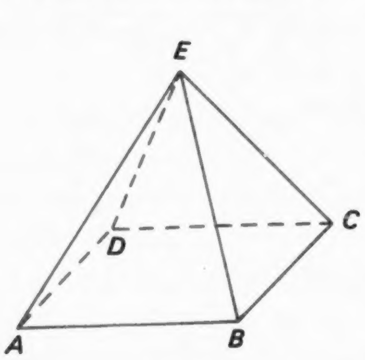

w


x


y

Next, let's consider a point in the air about two inches above rectangle *ABCD*. Suppose that we join with a segment each vertex of rectangle *ABCD* to the point in the air. The resulting geometric figure should resemble one of those pictured above. Which one? What is this figure called?

Class Discussion 9



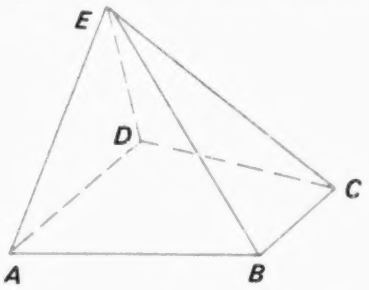
1. The pyramid pictured at the right has five *faces*. One of the faces is bounded by triangle *BCE*. Another face is bounded by quadrilateral *ABCD*. Name the polygons that bound the other three faces.
2. The pyramid pictured in exercise 1 has eight *edges*. An edge is a segment that is a side of the boundaries of two faces. For example, segment *AB* is an edge. Also, segment *EC* is an edge. Name the other edges of the pyramid.

3. The pyramid pictured in exercise 1 has five *vertices*. A vertex of a pyramid is an endpoint of an edge. Name the five vertices of the pyramid.

4. A closed geometric figure made up of flat faces bounded by polygons is called a *polyhedron*. Which of the figures pictured on page 45 are polyhedrons? Which ones are not polyhedrons?

5. What are the common names of figures **w**, **x**, and **y** pictured on page 45?

6. Name all angles formed by the edges of the polyhedron pictured at the right.



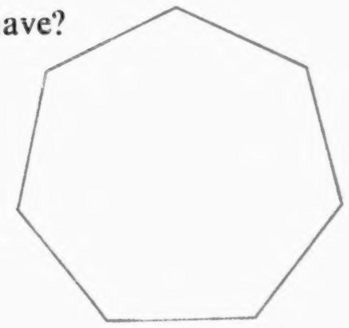
7. A *rectangular solid* is a polyhedron in which each face is bounded by a rectangle.

a. Which figure pictured on page 45 is a rectangular solid?
b. Make a sketch of a rectangular solid.
c. How many faces does a rectangular solid have?
d. How many edges does a rectangular solid have?
e. How many vertices does a rectangular solid have?

8. A *cube* is a rectangular solid in which each face is bounded by a square.

a. Make a drawing of a cube.
b. How many faces does a cube have?
c. How many edges does a cube have?
d. How many vertices does a cube have?

9. Think of a point about two inches in the air above the polygon pictured at the right. Now think of the polyhedron that is formed by joining with a segment each vertex of the polygon to the point in the air.



a. How many faces does this polyhedron have?
b. How many edges does this polyhedron have?
c. How many vertices does this polyhedron have?

It is hard to draw a picture of some polyhedrons because of the difficulty of representing three-dimensional figures on a flat surface. In the exercises that follow, some polyhedrons to which reference is made are pictured; but, because of the difficulty of making drawings, others are not. Therefore you will occasionally need to form a mental picture of a particular polyhedron.

All polyhedrons have faces, edges, and vertices. There is an interesting relationship that connects the number of faces, the number of edges, and the number of vertices of a polyhedron. In the exercises that follow you should be able to discover this relationship.

Exercises—9

1. Make a table like the one below.

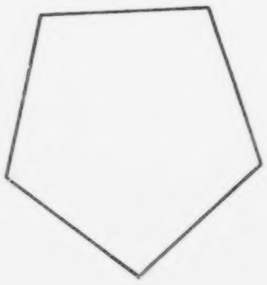
Figure	F = Number of Faces	V = Number of Vertices	E = Number of Edges
a			
b			
c			
d			
e			
f			
g			
h			
i			
j			
k			
l			
m			

a. Count the number of faces, vertices, and edges in figure **a** on page 45, and record the results in your table.
b. Count the number of faces, vertices, and edges in figure **b** on page 45, and record the results in your table.

c. Count the number of faces, vertices, and edges in figure **c** on page 45, and record the results in your table.

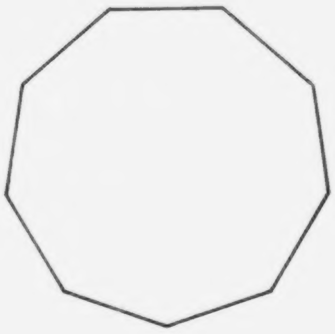
d. In row **d** of your table record the number of faces, vertices, and edges of the polyhedron discussed in exercise 9 of Class Discussion 9.

e. Consider the pentagon pictured at the right. Imagine that another pentagon is located in the air above it. Mentally join the corresponding vertices with segments. Determine *F*, *V*, and *E* for the polyhedron that is formed, and record this information in row **e**.



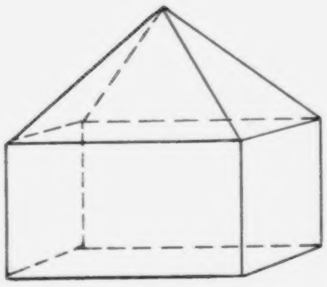
f. Now think of a single point located in the air above the pentagon represented in the last exercise. Mentally join each vertex of the pentagon with the point about which you are thinking. Determine *F*, *V*, and *E* for the resulting polyhedron and record this information in row **f**.

g. Follow the same instructions as in exercise 1e above, only this time consider a nine-sided polygon like the one pictured at the right. Record the numbers for *F*, *V*, and *E* in row **g**.



h. Follow the same instructions as in exercise 1f, only this time consider a nine-sided polygon. Record the numbers for *F*, *V*, and *E* in row **h** of your table.

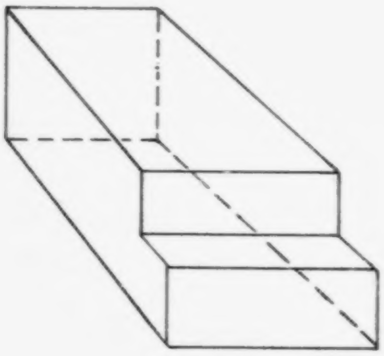
i. Count the faces, vertices, and edges of the polyhedron pictured at the right and record the results in row **i** of your table.



j. One polyhedron that is difficult to draw is the rhombicosidodecahedron. This polyhedron has 62 faces, 60 vertices, and 120 edges. Record this information in row **j** of your table.

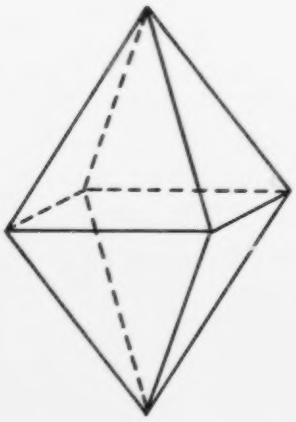
Before proceeding any further, carefully examine the information in your table for possible patterns.

k.



The drawing at the left shows a block of wood in which a rabbet joint has been cut. The drawing shows that the block has 8 faces and 12 vertices. Without counting, make a prediction as to the number of edges. Now count the edges to check your prediction and enter the information in row **k** of your table.

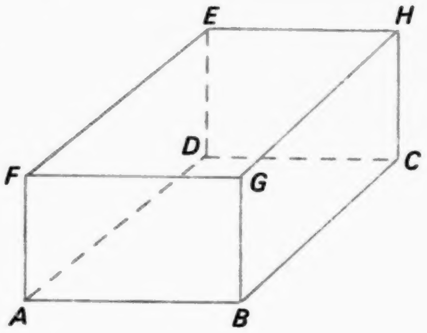
l. The polyhedron pictured at the right is an octahedron. It has 8 faces and 6 vertices Without counting, determine the number of edges. Check your conclusion by counting. Record the number of faces, vertices, and edges in row **l** of your table.



m. How do you think the number of faces, the number of vertices, and the number of edges of a polyhedron are related? Make up an example to test your answer. Record your results in row **m** of your table.

n. Write a formula that can be used to find the number of edges of a polyhedron if you know the number of faces and the number of vertices. Let E represent the number of edges; let F represent the number of faces; and let V represent the number of vertices.

2. **a.** Make a drawing of a rectangular solid and label it like the one pictured at the right.

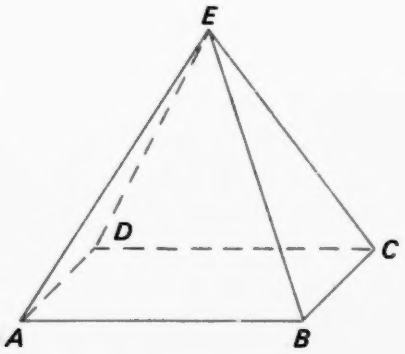


b. Recall that parallel lines are lines in the same plane that do not intersect. In the rectangular solid shown here name 3 edges that are included in lines that are parallel to the line that includes edge AB.

c. Recall that *skew lines* do not intersect but are not in the same plane. Name 4 edges that are included in lines that are skew to the line that includes edge AB.

d. Planes are parallel if they do not intersect. Referring to the rectangular solid shown above, name two faces that are in parallel planes.

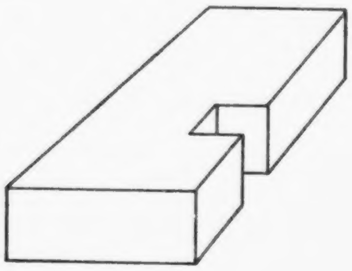
3. **a.** Draw a pyramid like the one pictured at the right and label it as indicated.



b. If possible, name a pair of edges that are included in parallel lines.

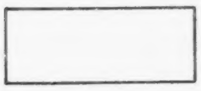
c. If possible, name a pair of faces that are in parallel planes.

4.

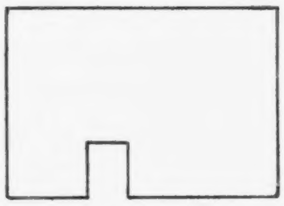


The drawing at the left shows a block of wood in which a groove has been cut. (Hidden lines are not shown.) If we were to look at this block from three different positions, which of the following sketches—**a**, **b**, or **c**—would represent the top view; the right side view; the front view?

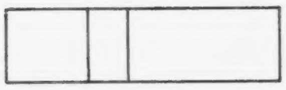
a



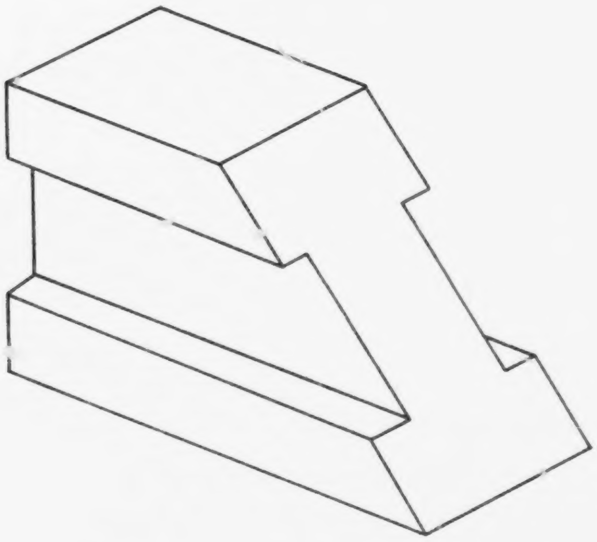
b



c



5. The drawing below shows a piece of a steel rail. (Hidden lines are not shown in the drawing.) Make a sketch of the top view, the right side view, and the back view of the figure.



10 The Angles of a Triangle

Class Discussion 10

1. Review exercise 16 on page 39 and explain what is meant by an "angle of a triangle."
2. Draw a triangle with one obtuse angle. A triangle with one obtuse angle is called an *obtuse triangle*.

3. Is it possible to draw a triangle that has two obtuse angles? Explain.

4. Draw a triangle that has one right angle. A triangle with one right angle is called a *right triangle*.

5. Can you draw a triangle with two right angles? Explain.

6. For each set of three angle measures listed below, try to draw a triangle having angles with the given measures.

- **a.** 60°, 60°, 60°
- **b.** 90°, 45°, 45°
- **c.** 75°, 75°, 30°
- **d.** 50°, 80°, 20°
- **e.** 70°, 60°, 80°
- **f.** 80°, 70°, 30°
- **g.** 30°, 50°, 80°
- **h.** 30°, 60°, 90°

7. Was it possible to draw a triangle for each set of three angle measures in exercise 6? For which sets were you able to draw triangles?

8. If you did your work accurately in exercise 6, you should have found that it is possible to draw triangles for cases a, b, c, f, and h. Find the sum of the angle measures in each case.

9. Referring again to exercise 6, you should have found that it is impossible to draw triangles for cases d, e, and g. Find the sum of the three angle measures given in each case.

10. Draw a triangle with three acute angles. Measure each angle and find the sum of the measures of the angles.

11. Draw a triangle ABC such that angle A has a measure of 40° and angle B has a measure of 60°.

- **a.** Without measuring the third angle, what do you predict its measure to be?
- **b.** Now measure the third angle with a protractor to test your prediction.

In each triangle that you drew in the exercises above, you should have found that the sum of the measures of the angles is approximately 180°

12. All measurements involve errors. If errors of measurement are taken into account, can you conclude that the sum of the measures of the angles in each triangle that you drew is *exactly* 180°?

Actually, there are three possibilities that would be consistent with the measurements that you made:

(1) The sum of the measures of the angles of a triangle may be exactly 180°.

(2) The sum of the measures of the angles of a triangle may be less than 180°, but the difference between 180° and the sum may be so small that it is covered up by errors of measurement.

(3) The sum of the measures of the angles of a triangle may be more than 180°, but the difference between 180° and the sum may be so small that it is covered up by errors of measurement.

13. If you measured the angles of one thousand triangles and in each case found their sum to be 180°, could you conclude from this evidence alone that the sum of the measures of the angles of every triangle is 180°? Can one prove by simply making measurements that the sum of the measures of the angles of every triangle is 180°?

On page 4 it was stated that there are three different mathematical models of space—the elliptic model, the Euclidean model and the hyperbolic model. All these statements can be proved:

(1) In the elliptic model, the sum of the measures of the angles of any triangle is greater than 180°.

(2) In the Euclidean model, the sum of the measures of the angles of any triangle is equal to 180°.

(3) In the hyperbolic model, the sum of the measures of the angles of any triangle is less than 180°.

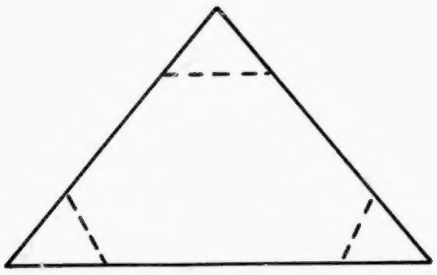
The proof of (2) is given in high school textbooks on geometry. The proofs of (1) and (3) are given in more advanced books on geometry. Since we are using the Euclidean model in this book, we shall assume the truth of (2) from now on.

In triangle ABC, let $m\angle A$, $m\angle B$, and $m\angle C$ represent the measures of angles A, B, and C respectively. Then

$$m\angle A + m\angle B + m\angle C = 180^\circ.$$

Exercises—10

1. a. Draw a large triangle on your paper. Now cut it out.
   b. Then cut off pieces at each vertex as shown in the picture.
   c. Use the pieces you obtained in exercise 1b to show that the sum of the measures of the angles of a triangle is 180°.

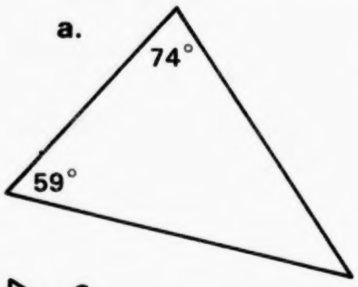


2. Use the formula $m\angle A + m\angle B + m\angle C = 180^\circ$ to complete the chart below.

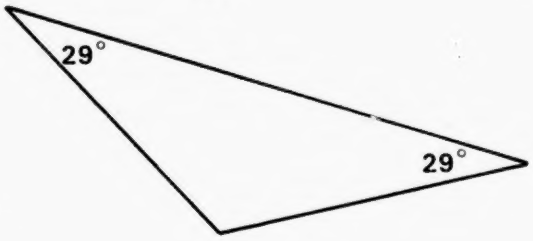
	$m\angle A$	$m\angle B$	$m\angle C$
Triangle I	70°	25°	85°
Triangle II	47°	39°	
Triangle III	23°	65°	
Triangle IV		59°	101°
Triangle V	165°		8°
Triangle VI		50½°	49½°

3. In each triangle pictured below, the measures of two of the angles are given. Determine, without using a protractor, the measure of the third angle.

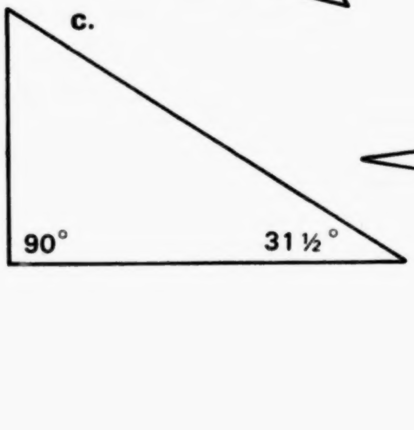
a.



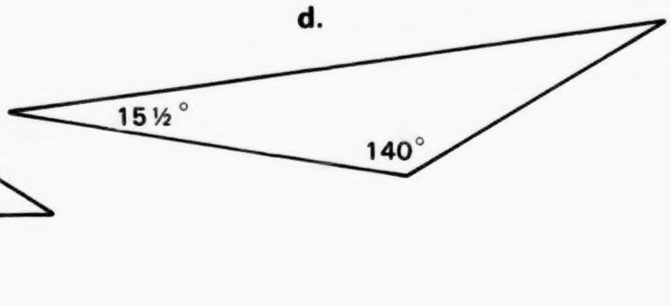
b.



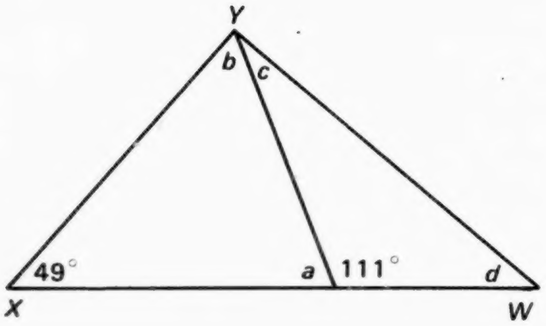
c.



d.



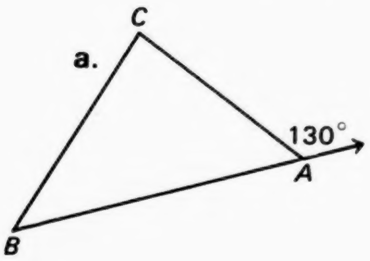
4. If angle XYW is a right angle, determine the measures of angle a, angle b, angle c, and angle d without using a protractor. (*Hint:* Find the measure of angle a first.)



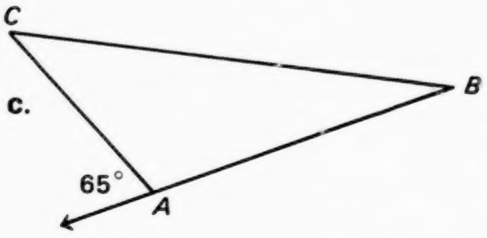
5. In each of the triangles pictured below, determine the measure of angle BAC and also the sum of the measures of angle B and angle C. Record your results in a chart like the one at the right. Do *not* use a protractor.

	$m\angle BAC$	$m\angle B + m\angle C$
a.		
b.		
c.		
d.		

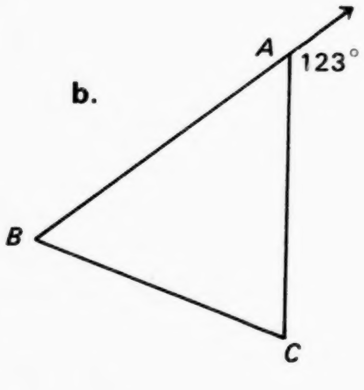
a.



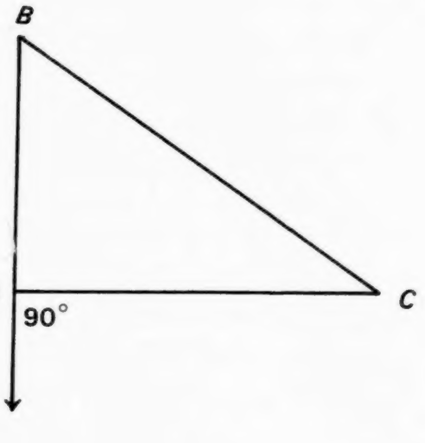
c.



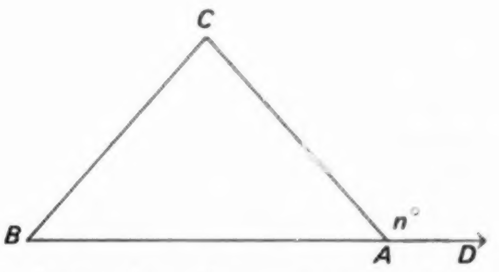
b.



d.

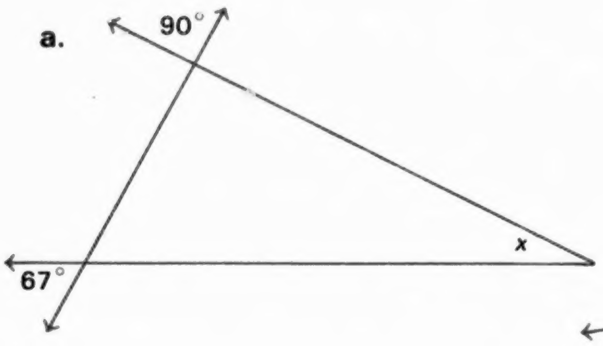


6. If angle *CAD* has a measure of *n* degrees, what is the sum of the measures of angle *B* and angle *C*?

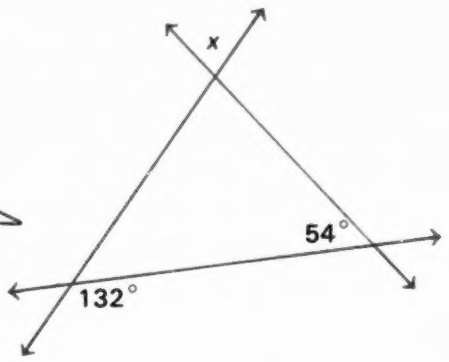


7. Without using a protractor, determine the measure of ∠ *x* in each figure.

a.



b.

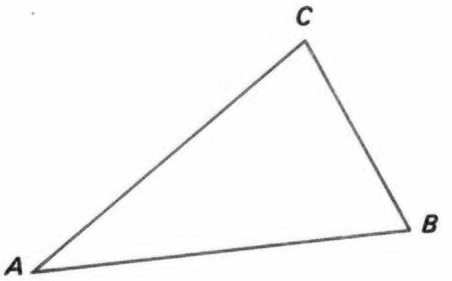


8. **a.** Draw a right triangle with an acute angle that has a measure of 35°.

b. What is the measure of the other acute angle?

9. Consider the triangle pictured at the right.

a. Draw a larger triangle whose angles have the same measures as those of triangle *ABC*. Label the new triangle *DEF*.

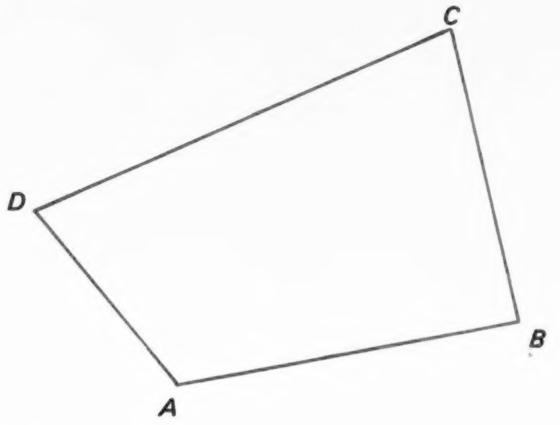


b. Triangles that have the same shape are said to be *similar*. Is triangle *ABC* similar to triangle *DEF*?

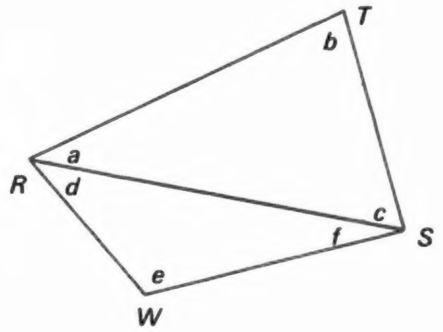
c. Draw another triangle that is similar to triangle *ABC*. Make it smaller than triangle *ABC*.

d. How many triangles are there that are similar to triangle *ABC*?

10. **a.** Measure each of the angles of quadrilateral *ABCD*.
b. Find the sum of the measures of the four angles.



11. A quadrilateral that has the same shape as the one in exercise 10 is shown at the right. Are the two quadrilaterals similar?



a. Into how many triangles does diagonal *RS* divide quadrilateral *RWST*?

b. $m\angle a + m\angle b + m\angle c =$ ___.

c. $m\angle d + m\angle e + m\angle f =$ ___.

d. $(m\angle a + m\angle b + m\angle c) + (m\angle d + m\angle e + m\angle f) =$ ___.

e. Does your answer to the last exercise represent the sum of the measures of the angles of quadrilateral *RWST*?

f. How does this result compare with the result you obtained by measurement in exercise 10?

12. Draw a quadrilateral *EFGH* with angles that have the same measures as the angles of quadrilateral *ABCD* in exercise

10, but do not make quadrilateral *EFGH* similar to quadrilateral *ABCD*.

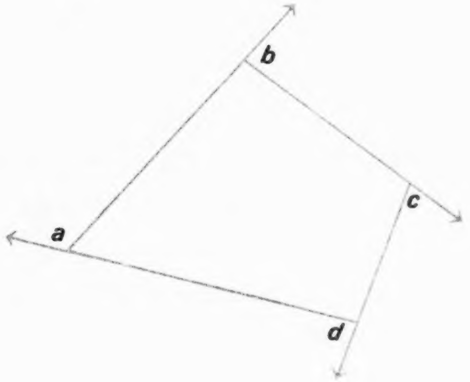
13. Make a sketch of a pentagon that has approximately the same shape as the one pictured at the right. Use the method suggested in exercise 11 to show that the sum of the measures of the angles of a pentagon is 540°.

14. Without measuring, determine the sum of the measures of the angles of the hexagon pictured at the right.

15. What is the sum of the measures of the angles of a polygon that has twenty sides?

16. Let N represent the number of sides of a polygon, and let S represent the sum of the measures of the angles of the polygon. Write a formula expressing S in terms of N.

17. Determine, without measuring, the sum of the measures of angle a, angle b, angle c, and angle d.



11 The Sides of a Triangle

In the last section you learned that in Euclidean geometry the sum of the measures of the angles of a triangle is 180°. Are the measures of the sides of a triangle also related in some way? The exercises that follow will guide you in finding an answer to this question.

Class Discussion 11

We saw on page 12 that we can use a pair of compasses to copy a segment of any given length. Compasses can therefore be used to draw a triangle when the lengths of the three sides are given. For example, to draw a triangle *ABC* whose three sides are to be 6 inches, 4 inches, and 3 inches, respectively, first make one side, say side *AB*, 6 inches long. Then, with *A* as center, and using a 4-inch radius, draw an arc. With *B* as center, and using a 3-inch radius, draw another arc. Let *C* be a point in the intersection of the two arcs. Segments *AB*, *AC*, and *BC* are the sides of the desired triangle *ABC*.

1. For each set of three lengths listed below try to draw a triangle with sides having the given lengths.
   - **a.** 4 inches, 3 inches, 2 inches
   - **b.** 5 inches, 4 inches, 2 inches
   - **c.** 3 inches, 2 inches, 1 inch
   - **d.** 4 inches, 2 inches, 1 inch
   - **e.** 3 inches, 4 inches, 5 inches
2. Is it possible to draw a triangle with sides having the lengths given in exercise 1d? Explain.
3. Is it possible to draw a triangle with sides having the lengths given in exercise 1c? Explain.
4. Is it possible to draw a triangle with sides having the lengths given in exercise 1e? What kind of triangle does it appear to be?

5. For each set of three lengths listed below, decide whether or not a triangle can be drawn with sides having the given lengths.

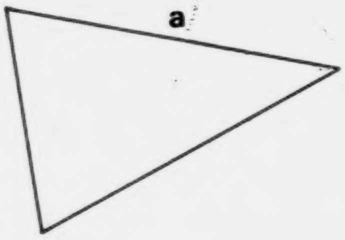
- **a.** 3 inches, 3 inches, 3 inches
- **b.** 1 foot, 2 feet, 2 feet
- **c.** 4 yards, 5 yards, 6 yards
- **d.** 2 inches, 1 inch, 1 inch
- **e.** 10 miles, 8 miles, 6 miles
- **f.** 3 feet, 3 feet, 4 feet
- **g.** 5 meters, 1 meter, 1 meter
- **h.** 15 inches, 2 inches, 3 inches
- **i.** 6 yards, 5 yards, 1 yard
- **j.** 4 feet, 4 feet, 4 feet

6. Make up a set of three lengths for which it is *impossible* to draw a triangle.

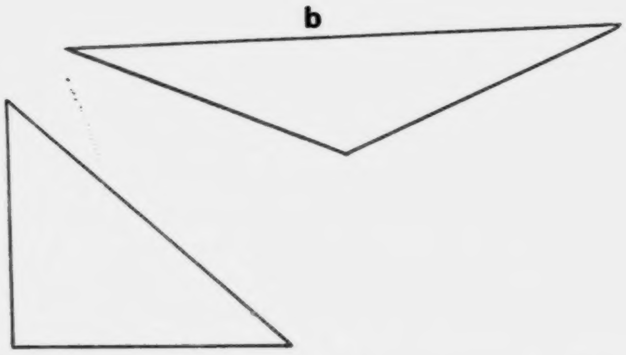
7. If at least two sides of a triangle have the same length, the triangle is called an *isosceles* triangle. Which sets of lengths in exercise 5 can be used to produce isosceles triangles?

8. If all sides of a triangle have the same length, the triangle is called an *equilateral* triangle. Which sets of lengths in exercise 5 can be used to produce equilateral triangles?

9. Use a pair of compasses to help you decide which of the triangles pictured below are isosceles triangles.



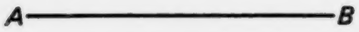
a



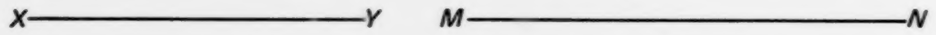
b

c

10. Use a pair of compasses to draw an equilateral triangle. Let each side have the same length as segment AB.



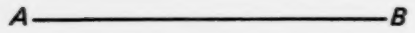
11. Use a pair of compasses to draw an isosceles triangle. Let the base of the triangle have the same length as segment XY. Let the other two sides have the same length as segment MN.



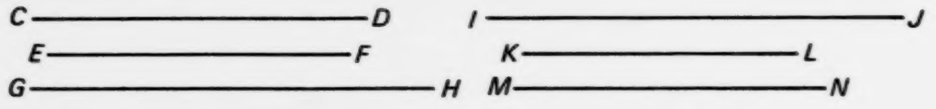
12. Use a pair of compasses and your ruler to help you draw a triangle whose sides have lengths of 2 inches, 2½ inches, and 3 inches.

In the exercises that follow, you are asked to make *geometric constructions*. In making a geometric construction you are allowed to use only two instruments, an unmarked straightedge and a pair of compasses. The "ground rules" do not permit you to use markings on a ruler, nor are you allowed to use a protractor.

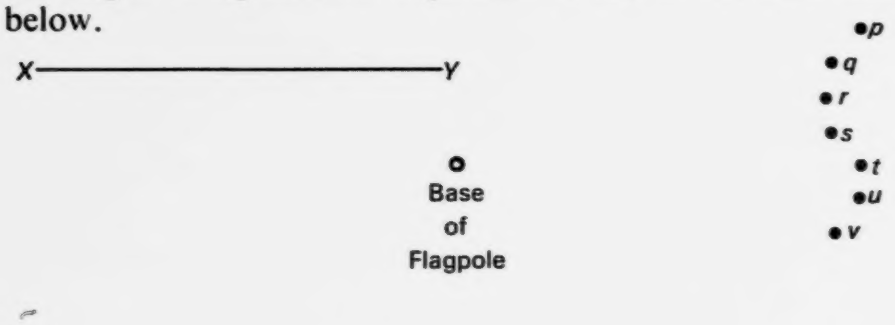
Exercises—11



1. Using a pair of compasses, determine which of the segments shown below are not as long as segment AB.



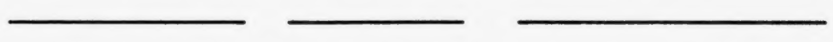
2. The plans for a garden call for each stepping-stone of a walk to be placed 30 feet from the base of a flagpole. Let the length of segment XY represent 30 feet in the scale drawing below.



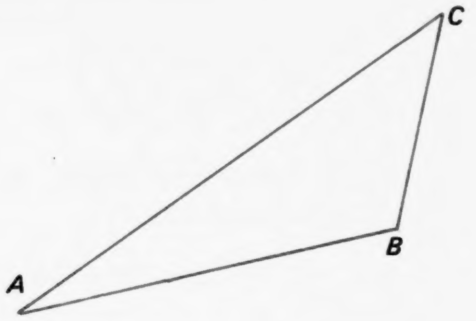
- **a.** Which of the stepping-stones are too far from the flagpole?
- **b.** Which of the stones are too close to the flagpole?
- **c.** Which stones are placed at the correct distance from the flagpole?
- **d.** If the complete path is laid out in accordance with the plans, what kind of figure will the stepping-stones form?

3. **a.** Construct a triangle in which each side is two inches long.
   b. What kind of triangle have you constructed?

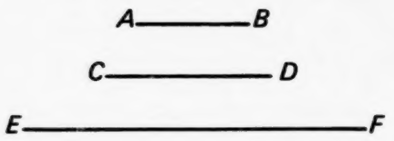
4. Construct a triangle whose sides have the same lengths as the segments pictured below.



5. Make a copy of triangle *ABC*. To make a copy of a given figure means to construct a figure that is congruent to the given figure. (See exercise 8d, page 12.)

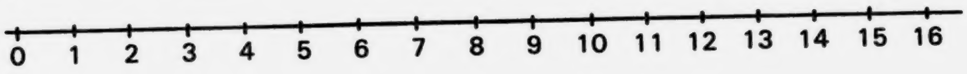


6. Consider the three segments *AB*, *CD*, and *EF*.

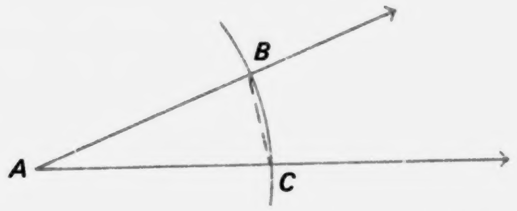


- **a.** If possible, construct a triangle with sides that have the same lengths as the given segments.
- **b.** What general statement can you make concerning the sum of the measures of any two sides of a triangle?

7. Use the scale below to construct a triangle whose sides have the following lengths: 3 units, 4 units, and 5 units; 5 units, 12 units, and 13 units. What appears to be true about each triangle that you constructed?



8. Copy angle A.

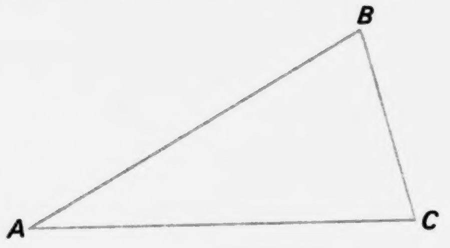


To copy angle A, first construct a triangle in which angle A is an *angle of the triangle*. (See exercise 16, page 39.) Then copy the triangle. For example, using A as center, and with any radius, draw an arc crossing the sides of angle A at B and C. Then draw segment BC. Now copy triangle ABC.

9. **a.** Draw an acute angle.
  b. Without using a protractor, construct another angle which has the same measure as the angle that you drew.

10. **a.** Draw an obtuse angle.
  b. Construct another angle that has the same measure as the obtuse angle that you drew.

11. **a.** Make an accurate copy of the isosceles triangle pictured below. The angle determined by the two sides that have the same length is called the *vertex angle* of the isosceles triangle. In triangle ABC, angle A is the vertex angle.



  b. Construct an isosceles triangle that is smaller than triangle ABC and in which the vertex angle has the same measure as angle A. Do the two triangles have the same shape? Are the two triangles similar? (See exercise 9b, page 56.)

c. Construct an isosceles triangle that is larger than triangle ABC and in which the vertex angle has the same measure as angle A. Is this larger triangle similar to triangle ABC?

d. How many different isosceles triangles can be drawn each having a vertex angle that has the same measure as angle A?

12. In this exercise, allow plenty of room for work.

a. Locate a point in the plane of your paper and label the point P.

b. Using the scale in exercise 7, locate two points that are 5 units from P.

c. Locate four more points that are 5 units from P.

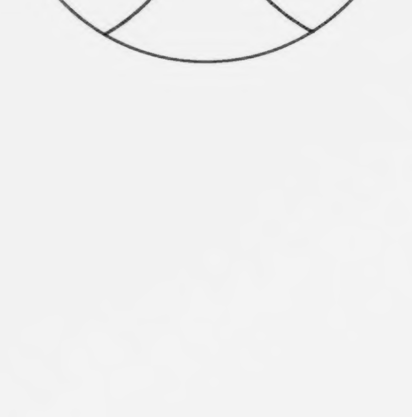
d. Locate all points that are 5 units from P. What geometric figure do you obtain?

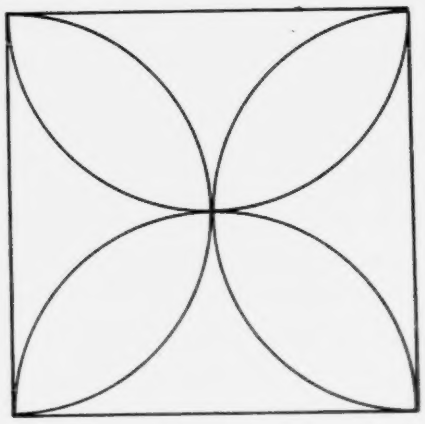
e. Locate a point that is 8 units to the right of P. Label this point Q.

f. Locate all points that are 5 units from Q.

g. Are there any points that are 5 units from P and also 5 units from Q? If so, identify these points in your drawing.

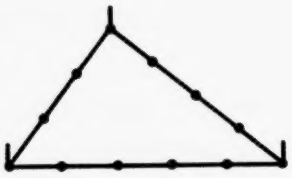
13. Use your compasses and straightedge to make a geometric design. Two examples are given below.

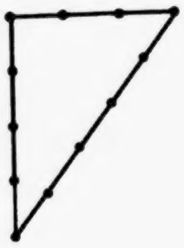




12 The Right Triangle

The people of ancient Egypt (about 3000 B.C.) were highly skilled in building and surveying. You may have seen pictures of the great pyramids and other structures that they built. Perhaps their achievements were due, in part, to an important mathematical discovery made by early man—a discovery with which the ancient Egyptians were acquainted. They knew that a loop of rope with 12 equally spaced knots could be positioned to form a right triangle. The rope was placed around three stakes as shown in the center diagram below.

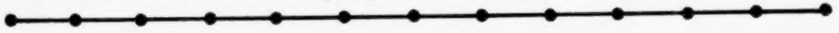




While the Egyptians knew that a loop with 12 knots worked, they apparently did not search for other ways of knotting a rope to produce a right triangle. We know today that loops with many different numbers of knots can be used.

Class Discussion 12a

1. Consider the knotted rope pictured below.

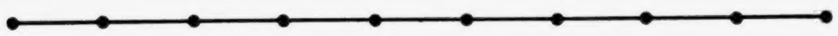


   a. Use a pair of compasses to help you see how to position the rope so that a triangle with sides of 3 units, 4 units, and 5 units is formed.
   b. Does the triangle that you constructed appear to be a right triangle?

c. Discuss all possible ways in which the rope pictured above can be positioned to form a triangle. A knot must be placed at a vertex.

d. Can the rope pictured above be positioned in any other way than is suggested in exercise 1a if a right triangle is to be formed?

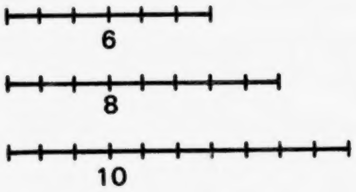
2. Another rope with equally spaced knots is pictured below. Can you position this rope so that a right triangle is formed? Remember, a knot must be placed at a vertex.



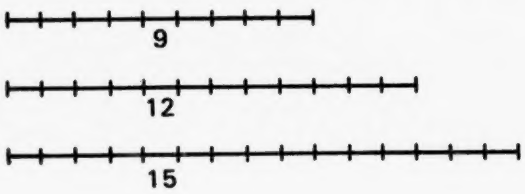
Exercises—12a

1. In each exercise below three segments are pictured. In each case construct a triangle in which the sides have the same lengths as the given segments.

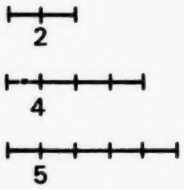
a. Construct a triangle having sides that are 6 units, 8 units, and 10 units in length. A triangle having sides with the given lengths is often referred to as a "6-8-10 triangle."



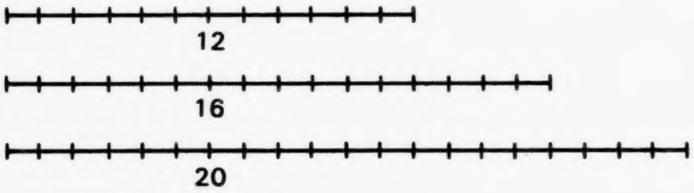
b. Construct a 9-12-15 triangle. (This means construct a triangle having sides that are 9 units, 12 units, and 15 units in length.)



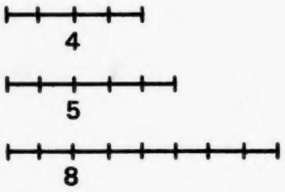
c. Construct a 2-4-5 triangle.



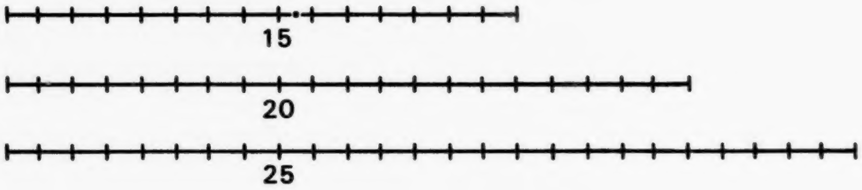
d. Construct a 12-16-20 triangle.



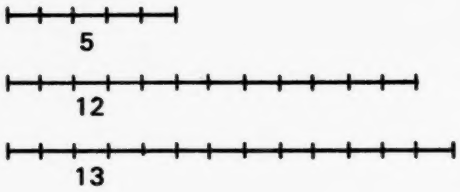
e. Construct a 4-5-8 triangle.



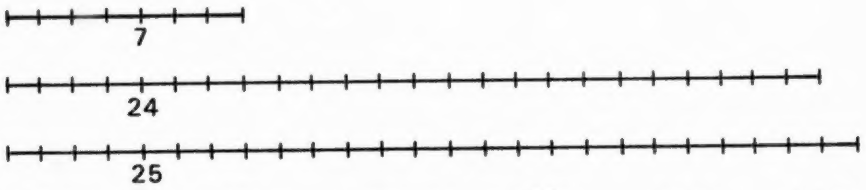
f. Construct a 15-20-25 triangle.



g. Construct a 5-12-13 triangle.



h. Construct a 7-24-25 triangle.



2. Some of the triangles that you constructed in exercise 1 are right triangles. Which combinations of lengths of sides in the accompanying list represent the lengths of the sides of right triangles?

3-4-5
6-8-10
9-12-15
2-4-5
12-16-20
4-5-8
15-20-25
5-12-13
7-24-25

3. Perhaps you noticed a pattern in the lengths of the sides of some of the right triangles that you constructed in exercise 1. Examine the selections that you made from the list in exercise 2. Then complete the table below.

Lengths of the Sides of a Right Triangle		
Shortest Side	Second Side	Longest Side
6	8	10
15	20	
18	24	
21	28	
30	40	
300	400	

4. Make a scale drawing to decide whether or not the following combination of lengths of sides determines a right triangle.

8-15-17

Class Discussion **12b**

1.

Shortest Side	Second Side	Longest Side
3	4	5
6	8	10
5	12	13
8	15	17

Each set of three lengths listed in the table above represents the lengths of the sides of a right triangle. To help you see how the three lengths in each set are related, a table listing the squares of the lengths in each set is given below.

Shortest Side Squared	Second Side Squared	Longest Side Squared
3^2	4^2	5^2
6^2	8^2	10^2
5^2	12^2	13^2
8^2	15^2	17^2

a. You know that $3^2 = 9, 4^2 = 16$, and $5^2 = 25$. How are the numbers 9, 16, and 25 related?

b. You know that $6^2 = 36, 8^2 = 64$, and $10^2 = 100$. How are the numbers 36, 64, and 100 related?

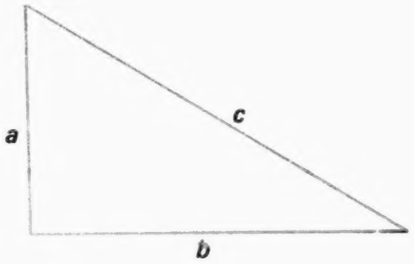
c. Is it true that $5^2 + 12^2 = 13^2$?

d. Is it true that $8^2 + 15^2 = 17^2$?

e. On the basis of your answers to exercises 1a through 1d, tell how the squares of the lengths of the sides of a right triangle are related. Check to see whether or not this relationship holds for the entries in the table in exercise 3 above.

2. Let a, b, and c represent the lengths of the sides of a triangle, where c is the length of the longest side.

In Euclidean geometry it can be proved that if $a^2 + b^2 = c^2$, then the triangle is a right triangle. Is a triangle in which $a = 48$, $b = 55$, and $c = 73$ a right triangle?



Summary—12b

You have found that it is not necessary to construct a triangle to determine if it has a right angle. Instead, you can make use of a formula. As stated above, if a, b, and c represent the lengths of the sides of a triangle and $a^2 + b^2 = c^2$, then the triangle is a right triangle.

The converse of the last statement is also true. That is, if we have a right triangle, then

$$a^2 + b^2 = c^2.$$

This formula is usually attributed to a man who lived in Greece some 2,500 years ago. The man's name is Pythagoras, and the formula is often referred to as the *rule of Pythagoras*.

Exercises—12b

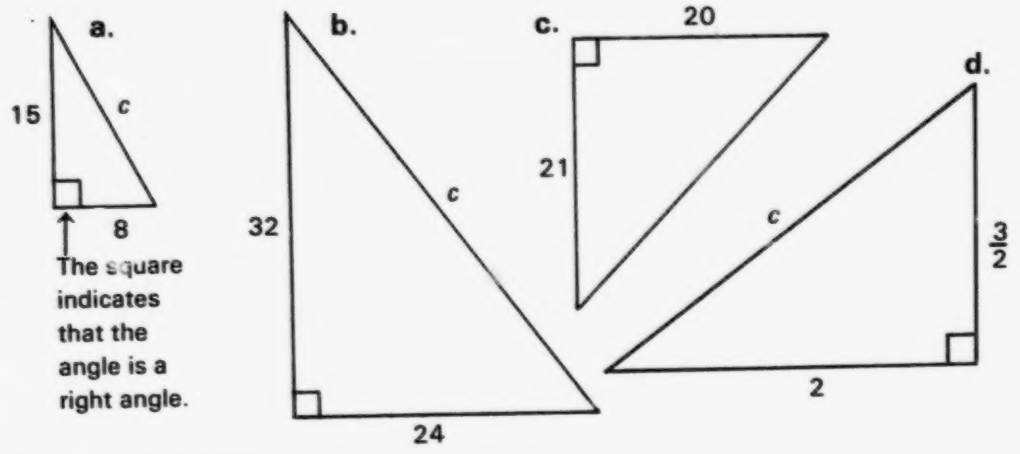
1. Use the formula $a^2 + b^2 = c^2$ to determine whether or not triangles with sides having the following lengths are right triangles.

a	b	c
24	10	26
11	15	18
8	11	14
1	$\frac{3}{4}$	$\frac{5}{4}$

2. Copy and complete the table below.

c	4		17		29	$\frac{5}{2}$	40
c^2		25		400			

3. Use the rule of Pythagoras to find c^2 for each of the right triangles pictured below. Then use a table of squares, or use trial and error, to find c.



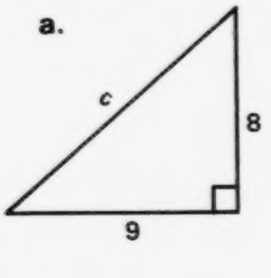
The square indicates that the angle is a right angle.

4. In each exercise below you will find that the number represented by c is between two whole numbers. In each case indicate which two whole numbers are nearest c.

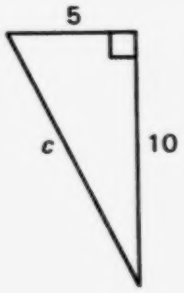
a. If c^2 is 12, then c is between__and__.
b. If c^2 is 57, then c is between__and__.
c. If c^2 is 93, then c is between__and__.
d. If c^2 is 171, then c is between__and__.

5. In each right triangle below use the rule of Pythagoras to find c^2; then find an approximation of c. Give your answer to the nearest whole number.

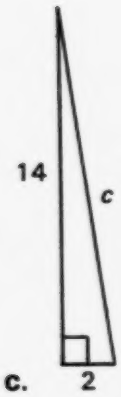
a.



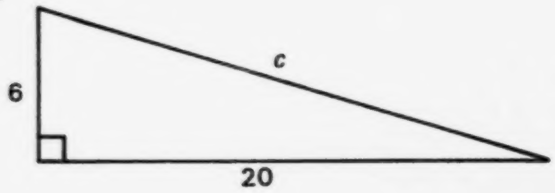
b.



c.



d.

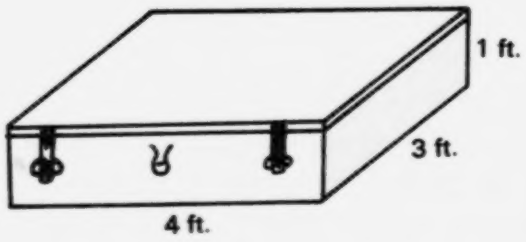


6. A rectangle is 13 inches long and 7 inches wide. Find the length of a diagonal of the rectangle to the nearest inch.

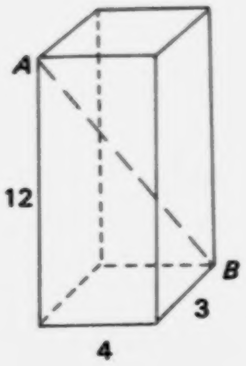
7. Mr. Jones owns a rectangular lot which is 120 feet long and 90 feet wide. How long is the path that runs from one corner of the lot to the opposite corner?

8. Leaving Portland, the S.S. *Teakettle* sails 30 miles north and then sails 50 miles west. How far is the ship from Portland? Express your answer to the nearest mile.

9. Mr. Smith has a trunk 4 feet long, 3 feet wide, and 1 foot deep. Find the length of the longest fishing rod that can be put in the trunk.



10. Find the length of the diagonal *AB* of the rectangular prism pictured below.



13 The Rhombus and Constructions

Class Discussion 13

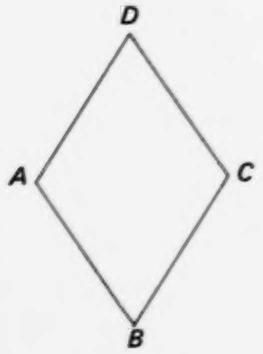


Fig. 23

The diamond-shaped figure shown above is a quadrilateral whose four sides all have the same length. A quadrilateral whose four sides all have the same length is called a *rhombus*. In this section we shall discover some properties of a rhombus that make it useful for carrying out some important geometric constructions.

Two vertices of a quadrilateral are called *opposite vertices* if they are not endpoints of the same side. The segment joining opposite vertices of a quadrilateral is a diagonal of the quadrilateral.

1. Name one pair of opposite vertices of the quadrilateral shown in Figure 23.
2. Name a second pair of opposite vertices of the quadrilateral shown in Figure 23.

To construct a rhombus, begin by locating two points that are to be opposite vertices of the rhombus. In Figure 24, points P and Q have been chosen as opposite vertices. Draw segment PQ. Choose some convenient length as radius and, with P and Q as centers, draw arcs of circles. Label the points of intersection of these arcs R and S. Draw segments PR, RQ, QS, and SP. The four-sided figure $PSQR$ is a rhombus, and segment PQ is one of its diagonals.

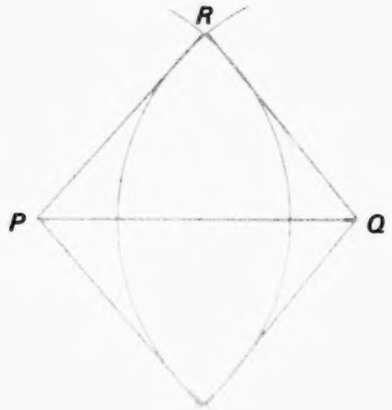


Fig. 24

3. Locate points *P* and *Q* so that segment *PQ* has a given length. Repeat the rhombus construction several times, each time using a smaller radius when you draw the arcs with *P* and *Q* as centers.
   - **a.** Can you complete the construction with any choice of radius, no matter how small?
   - **b.** How long must the radius be to guarantee that the arcs drawn with centers *P* and *Q* will intersect at two points?

Using two points *P* and *Q* as opposite vertices, construct a rhombus *PSQR* in the manner described above. Draw diagonals *PQ* and *RS*, and label the point at which they intersect *M*.

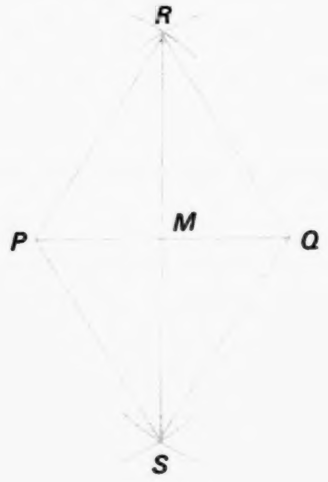


Fig. 25

4. Measure segments *PM* and *MQ*. How do they compare in length?
5. Measure segments *RM* and *MS*. How do they compare in length?

The point *M*, which divides segment *PQ* into two congruent segments, is called the *midpoint* of segment *PQ*. Any line (or segment) which crosses a given segment at its midpoint is called a *bisector* of the given segment. Hence, line *RS* is a bisector of segment *PQ*. Also, segment *RS* is a bisector of segment *PQ*.

6. Name a line that is a bisector of segment *RS*. Is segment *PQ* a bisector of segment *RS*?

Segment *RS* bisects segment *PQ*, and segment *PQ* bisects segment *RS*. It is possible to prove that in any rhombus the diagonals bisect each other.

7. Measure angle *PMR* and angle *RMQ* with your protractor. How do the measures of these two angles compare?
8. Since the linear pair of angles *PMR* and *RMQ* are congruent, what can we say about the way in which lines *PQ* and *RS* are related to each other?
9. Construct several more rhombuses and draw their diagonals. Compare the measures of the angles determined by the two diagonals in each case.

The exercises above suggest that the diagonals of a rhombus are perpendicular to each other. It is possible to prove that this is true in every rhombus. When we say that two segments (diagonals in this case) are perpendicular to each other, we mean that they intersect and are included in lines that are perpendicular. See exercise 6, page 25, for a discussion of perpendicular lines.

10. Draw a segment *AB*, and let point *M* be its midpoint. Draw a line through *M*. Draw another line through *M*. How many different lines may be drawn through *M*?
11. How many lines are there that bisect segment *AB*?
12. How many of these lines are also perpendicular to segment *AB*?

A line (or a segment) that bisects a given segment and is also perpendicular to it is called the *perpendicular bisector* of the given segment. Each diagonal of a rhombus is the perpendicular bisector of the other.

13. Using Figure 25, measure angle *RPM* and angle *MPS*, and compare their measures.

14. Compare the measures of angles *RQM* and *SQM*.

15. Compare the measures of angles *PRM* and *QRM*.

16. Compare the measures of angles *PSM* and *QSM*.

Angle *RPS* is an angle of rhombus *PSQR*. The diagonal *PQ* divides angle *RPS* into two congruent angles. So *PQ* is the bisector of angle *RPS*. In exercise 3 on page 34 it was stated that the bisector of an angle is a ray. When we say that a segment (in this case, a diagonal) bisects an angle, we mean that one endpoint of the segment is at the vertex of the angle and the other endpoint is in the ray that is the bisector of the angle.

It is possible to prove that in every rhombus each diagonal bisects the two opposite angles whose vertices it joins.

17. Construct a rhombus and examine a pair of opposite sides. Do they appear to be parallel? It can be proved that in any rhombus the opposite sides are parallel.

Summary—13

1. A rhombus is a quadrilateral whose four sides are all congruent.

2. The diagonals of a rhombus bisect each other.

3. The diagonals of a rhombus are perpendicular to each other.

4. Each diagonal of a rhombus is the perpendicular bisector of the other.

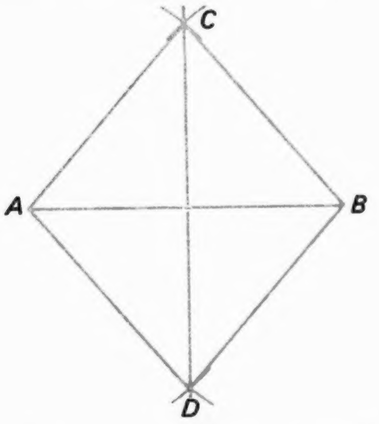
5. Each diagonal of a rhombus bisects the two opposite angles whose vertices it joins.

6. The opposite sides of a rhombus are parallel.

Exercises—13a

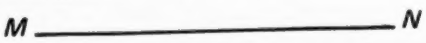
Construction of the perpendicular bisector of a segment

1. Let AB be any segment. If we want to construct its perpendicular bisector, all we have to do is construct a rhombus in which segment AB is one diagonal. Then the endpoints of the other diagonal determine the perpendicular bisector of segment AB. In the construction shown below, the line through C and D is the perpendicular bisector of segment AB.



To complete the construction of line CD, is it necessary to draw the sides of rhombus $ADBC$?

2. a. Make a copy of segment MN.



b. Construct the perpendicular bisector of segment MN.

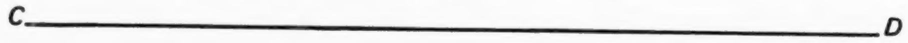
3. In this exercise you are *not* allowed to use a pair of compasses.
   a. Draw a segment $3\frac{5}{16}$ inches long.
   b. Find the midpoint of this segment *using only your ruler*.
4. a. Draw another segment $3\frac{5}{16}$ inches long.
   b. Find the midpoint of this segment by *constructing* the perpendicular bisector of the segment as described in exercise 1.
5. Allowing plenty of room for work, draw a segment AB that is 4 inches long.

- **a.** Find all points which are 3 inches from both *A* and *B*; 2 ¾ inches; 2 ½ inches; 2 ¼ inches; 2 inches.
- **b.** Why is it impossible to find any points that are 1½ inches from both *A* and *B*?

6. Locate two points in the plane of your paper and label them *X* and *Y* respectively.

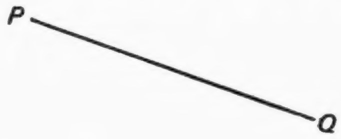
- **a.** Find two points each of which is the same distance from both *X* and *Y*.
- **b.** Find two more points each of which is the same distance from both *X* and *Y*.
- **c.** How many points are there each of which is the same distance from both *X* and *Y*?
- **d.** What geometric figure is suggested by all points each of which is the same distance from both *X* and *Y*?

7. a. Make a copy of segment *CD*.

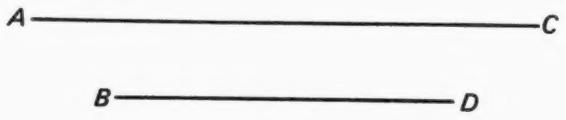


- **b.** Construct three lines that divide segment *CD* into four congruent segments.
- **c.** By extending the procedure you used in exercise 7b, divide segment *CD* into 8 congruent segments; 16 congruent segments.

8. Make a copy of segment *PQ* and draw three lines that pass through its midpoint.

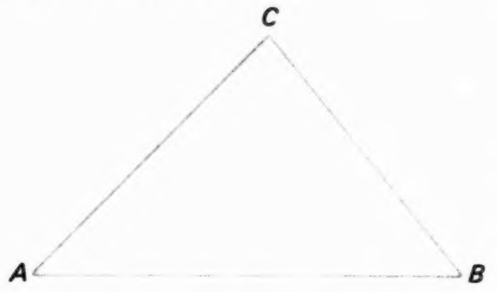


9. Construct a rhombus in which the two diagonals have the same lengths as segments *AC* and *BD*.



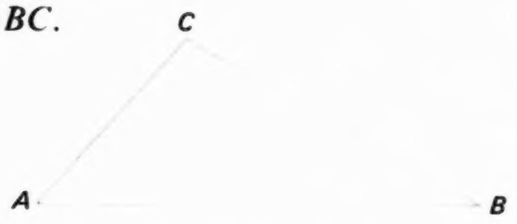
Will your rhombus be congruent to those constructed by your classmates?

10. a. Copy the triangle pictured below.



b. Construct the perpendicular bisector of each side. If this construction is done correctly, the three perpendicular bisectors will all meet in a point.

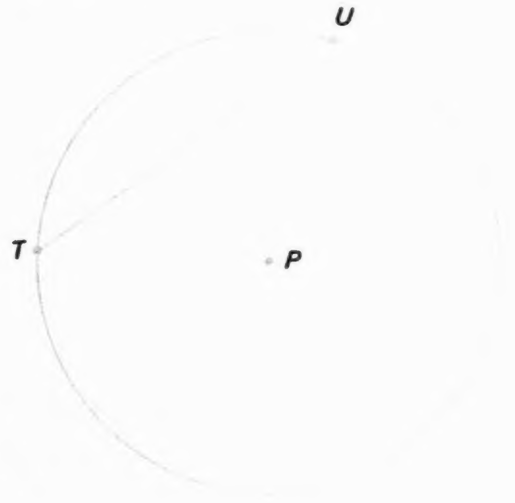
11. a. Copy triangle ABC.



b. Find the midpoint of each side by finding the perpendicular bisector of each side.

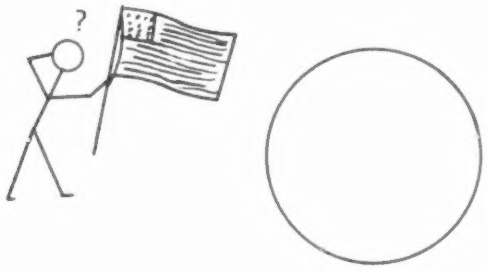
c. Draw a segment from the midpoint of each side to the opposite vertex. Such a segment is called a *median* of the triangle. If the construction is done correctly, the three medians will pass through the same point.

12.



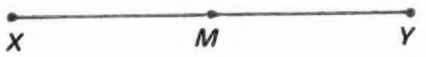
Points T and U are in a circle whose center is P. Construct a rhombus in which points T, P, and U are vertices. Is point P in the perpendicular bisector of segment TU?

13. Mr. McKay had a circular-shaped block of wood which he wanted to use as a stand for a flagpole. Unfortunately, the center was not marked, and he did not know where to drill a hole for the flagpole. Show how to locate the center of the circular-shaped block using only a straightedge and a pair of compasses.

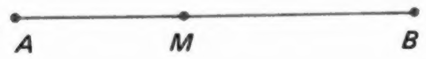


Exercises—13b

Construction of the perpendicular to a line at any point in the line

1.


In the diagram at the left, M is the midpoint of segment XY. Copy the diagram and construct the perpendicular bisector of segment XY.

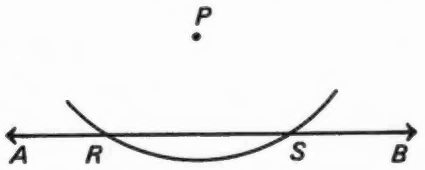
2.


In the diagram at the left, M is not the midpoint of segment AB. Copy the diagram and construct a segment AD on AB that has M as its midpoint. Then construct a rhombus in which segment AD is a diagonal.

3. Draw a line EF and let P be any point in the line. Taking your cue from exercise 2, construct a line through P that is perpendicular to line EF.

Exercises—13c

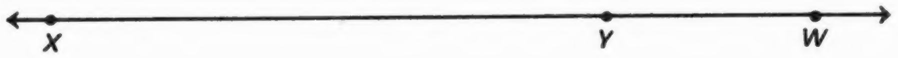
Construction of the perpendicular to a line from a point that is not in the line

1. 

In the diagram at the left point P is not in line AB. Copy the diagram. Using P as center, draw an arc of a circle that intersects line AB in two points. Label the points of intersection R and S. Complete a rhombus in which segments PR and PS are sides.

2. Let line XY be given, and let A be any point that is not in line XY. Taking your cue from exercise 1, construct a line through A that is perpendicular to line XY.

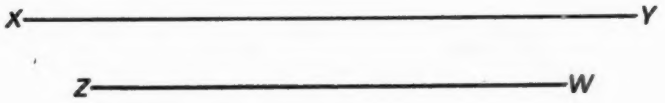
3. Copy the figure below.



   a. Through point Y construct a line perpendicular to line XW.
   b. Through point X construct a line perpendicular to line XW.
   c. Through point W construct a line perpendicular to line XW.
   d. What conclusion can you state concerning the three lines that you constructed?

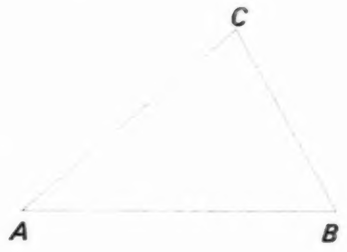
4. Construct a square in which each side has the same length as segment AB.

5. Construct a rectangle whose length has the same measure as segment XY and whose width has the same measure as segment ZW.



6. **a.** Make a copy of triangle ABC.

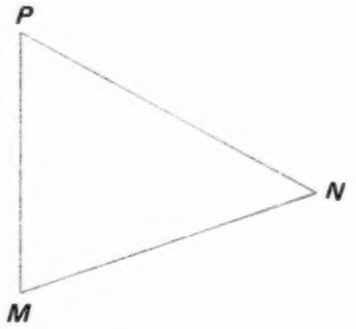
b. Through point C construct a line perpendicular to the line that includes side AB. Label the point of intersection X.



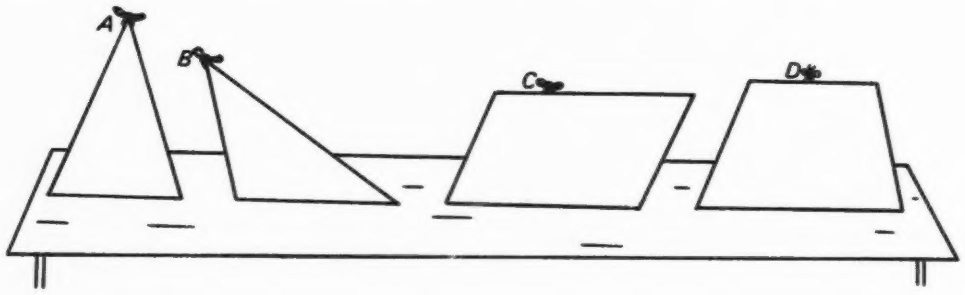
c. Segment CX is called an *altitude of the triangle*. An altitude of a triangle is a segment that has a vertex of the triangle as one endpoint and a point in the line that includes the side opposite the vertex as the other endpoint. The altitude is perpendicular to the line that includes the side opposite the vertex. Does triangle ABC have other altitudes besides the altitude CX? Explain.

7. Make a copy of triangle MNP, and construct the three altitudes of the triangle.

If the three altitudes are constructed correctly, they will meet in a single point.



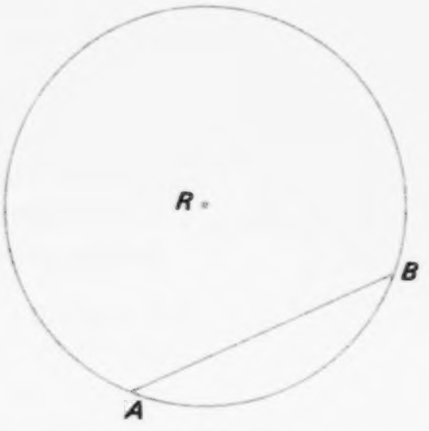
8. Below is a side view of several blocks of wood resting on a tabletop. A fly lands on the top of each block.



a. In which case is the fly at the greatest distance from the tabletop?

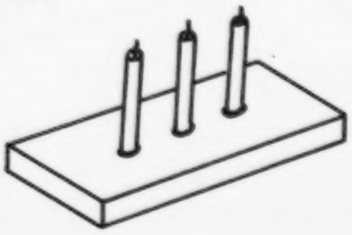
b. Draw four figures that are larger but have approximately the same shapes as those pictured above. Construct the line segments that you would need to measure to find the distance of the fly from the tabletop in each case.

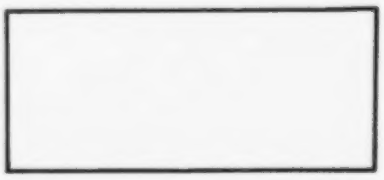
9. A segment having its endpoints in a circle is called a *chord* of the circle. In the diagram below, segment AB is a chord of the circle. Point R is the center of the circle.



a. Copy the diagram on your paper.
b. Construct a line through R perpendicular to chord AB.
c. Construct the perpendicular bisector of chord AB.
d. Do the two constructions in exercises 9b and 9c give you the same line?

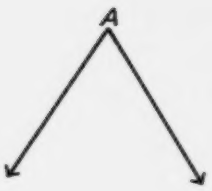
10. Paul has a block of wood and wants to make a holder for three candles. He wants the candles to be evenly spaced on the board, but he does not know how to find the points at which to drill the holes.



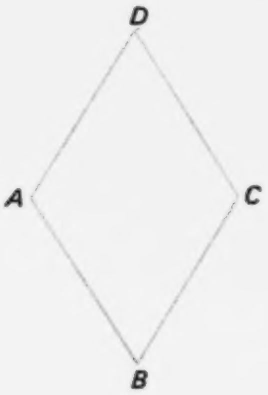


Suppose that the rectangle on the right above represents the top of the block of wood. Trace the rectangle on your paper and see if you can discover a construction for locating the points at which the holes should be drilled.

11. **a.** Draw an angle on your paper that is congruent to angle A.
b. Now construct a rhombus in which the angle that you drew in exercise 11a is an angle of the rhombus.



12.



Consider the rhombus pictured at the left.

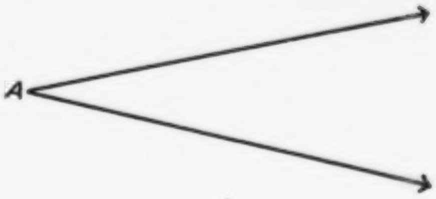
a. Using a straightedge and a pair of compasses, construct a rhombus that is congruent to the given rhombus.

b. Construct another rhombus that is larger but similar to the given rhombus.

Exercises—13d

Construction of the bisector of an angle

1. Copy angle A. Then construct a rhombus in which angle A is an angle of the rhombus. Draw the diagonal of the rhombus that has vertex A as one of its endpoints.

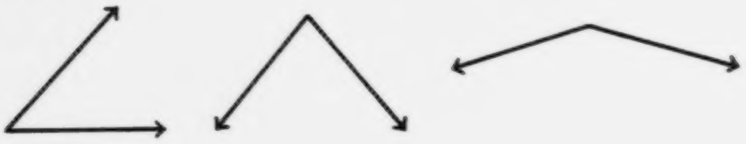


2. Draw any angle. Taking your cue from exercise 1, bisect the angle. (Recall that the bisector of an angle is a ray; see exercise 3, page 34, and the comment after exercise 16, page 76.)

3. **a.** Draw an acute angle whose measure is approximately 60°.
b. Construct the bisector of the angle.

4. **a.** Draw an obtuse angle.
b. Construct the bisector of the angle.

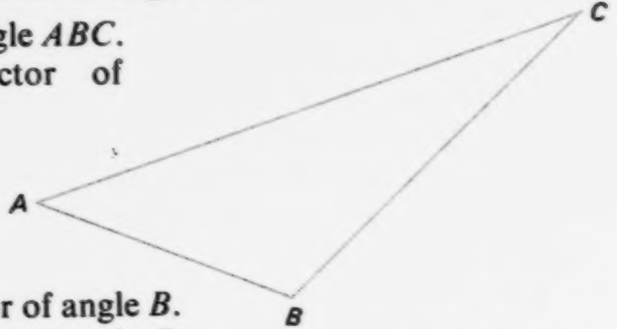
5. a. Draw angles that have approximately the same measures as those shown below, and construct the angle bisector of each.



b. How many bisectors does an angle have?

6. a. Make a copy of triangle ABC.

b. Construct the bisector of angle A.

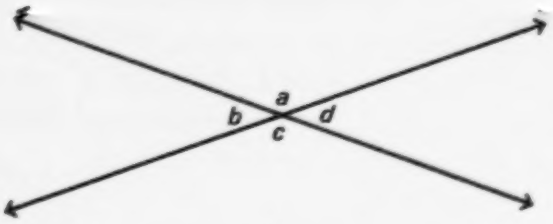


c. Construct the bisector of angle B.

d. Construct the bisector of angle C.

If the constructions are made correctly, the three angle bisectors should meet in one point.

7. a. Draw two intersecting lines as shown below.



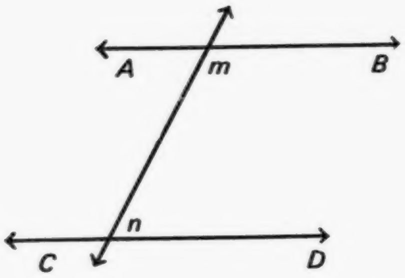
b. Construct the bisector of angle a.
Construct the bisector of angle b.
Construct the bisector of angle c.
Construct the bisector of angle d.

c. Describe the position that the bisectors of angle a and angle d seem to have with respect to each other.

d. Describe the position that the bisectors of angle b and angle d seem to have with respect to each other.

e. Do the bisectors of angle b and angle d appear to be opposite rays?

8. In the figure shown at the right, line AB is parallel to line CD.



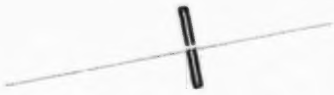
- **a.** Copy the figure.
- **b.** Construct the bisector of of angle m.
- **c.** Construct the bisector of angle n.
- **d.** Describe the position that these bisectors seem to have with respect to each other.

9. Follow the directions below carefully.

- **a.** Draw an angle having a measure of approximately 60° and label the angle ABC.
- **b.** Construct the bisector of angle ABC.
- **c.** Select any point in the angle bisector and call it P.
- **d.** Through point P construct a line perpendicular to side BA. Name the point of intersection X.
- **e.** Through point P construct a line perpendicular to side BC. Name the point of intersection Y.

If your constructions are done accurately, segment PX should have the same length as segment PY.

10. Calvin invented an instrument that can be used to bisect an angle. The instrument consists of a piece of stiff wire fastened to a strip of wood in such a way that the wire indicates the position of the perpendicular bisector of the strip of wood. The instrument is pictured below.

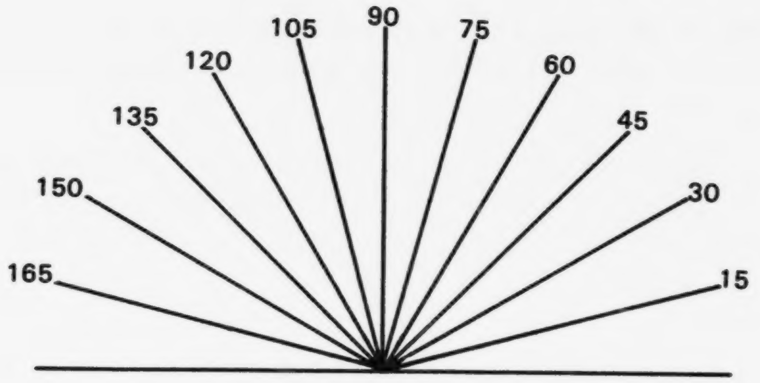


Make an instrument like Calvin's and use it to find the bisector of an angle.

11.
- **a.** Construct a right angle.
- **b.** Construct the bisector of the right angle.
- **c.** What is the measure of each of the newly formed angles?
- **d.** Using the same figure, construct an angle having a measure of 22½°.

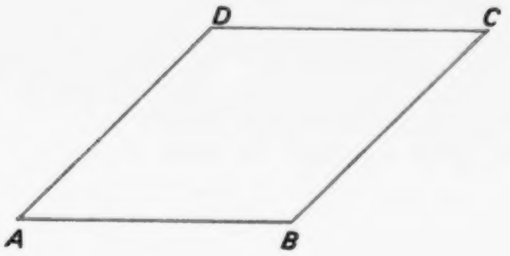
e. Again using the same figure, construct an angle having a measure of 67 ½ °.

12. By using only a straightedge and a pair of compasses, construct a protractor that can be used to measure angles that are multiples of 15°. The sketch illustrates what is required.

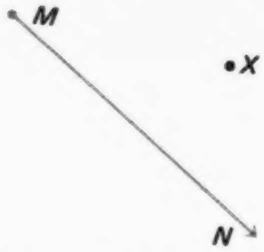


Exercises—13e

Constructing parallel lines



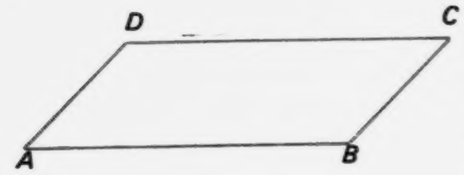
1. If quadrilateral $ABCD$ is a rhombus, name two sides that are parallel.



2. Make a copy of the diagram at the right. Draw segment MX. On ray MN, construct a segment MY congruent to segment MX. Then construct a rhombus in which M, X, and Y are vertices.

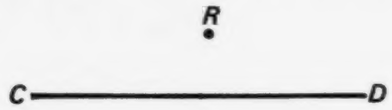
3. Given a line AB and a point P that is not in line AB, take your cue from exercise 2 and construct a line through P that is parallel to line AB.

4. In quadrilateral *ABCD*, the opposite sides *AB* and *DC* are congruent, and the opposite sides *AD* and *BC* are congruent. But quadrilateral *ABCD* is not a rhombus. Why not?



It can be shown that if the opposite sides of a quadrilateral are congruent, then they are parallel, and hence the quadrilateral is a parallelogram.

5. Let *R* be a point that is not in segment *CD*. Draw segment *CR*. In this figure, segment *CR* is not congruent to segment *CD*. Construct a parallelogram in which *R*, *C*, and *D* are vertices.



6. Given a line *AB* and a point *P* that is not in line *AB*, take your cue from exercise 5 and construct a line through *P* that is parallel to line *AB*.

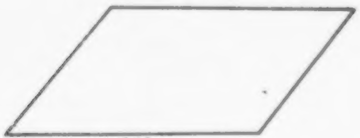
7. Recall that a parallelogram is a quadrilateral with opposite sides parallel.

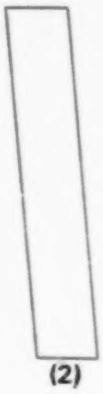
a. According to this definition, which of the drawings below represent parallelograms?

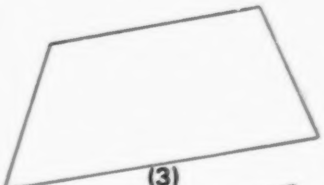
b. Is a rectangle a parallelogram?

c. Is a square a parallelogram?

d. Is a rhombus a parallelogram?


(1)


(2)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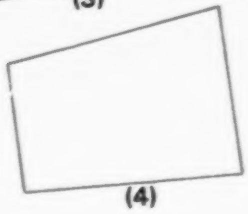
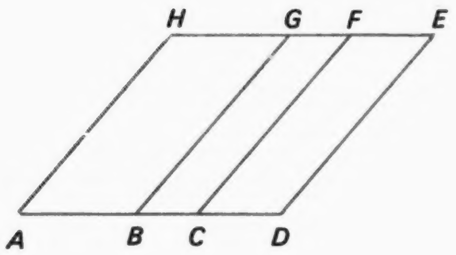
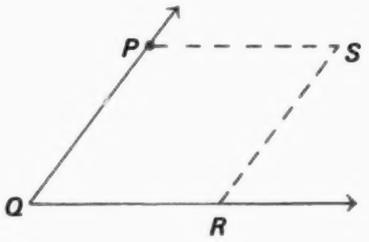

(4)

8. Name all parallelograms in the figure shown at the right.

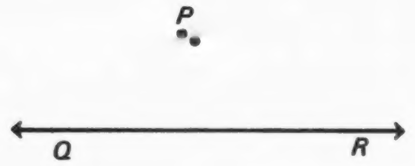


9. **a.** Draw an angle Q and locate a point P in one side of the angle as shown in the picture.

b. Now complete the construction of a rhombus. The rhombus should be labeled like the one in the picture.

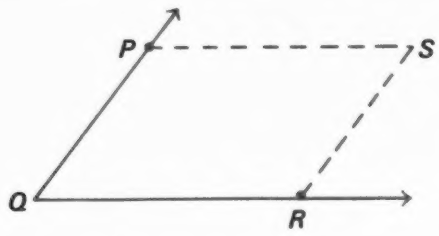


10. Make a copy of the diagram below. Through point P construct a line parallel to line QR.

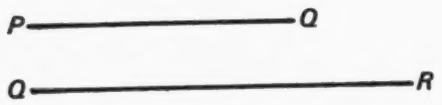


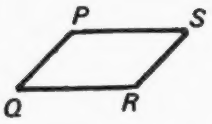
11. **a.** Draw an angle Q. Locate a point P in one side of the angle and a point R in the other side.

b. Now complete the construction of parallelogram $PQRS$.



12. **a.** Construct a parallelogram $PQRS$. Sides PQ and QR of the parallelogram should have the same lengths as the segments below. The completed figure should be labeled like the one shown below.



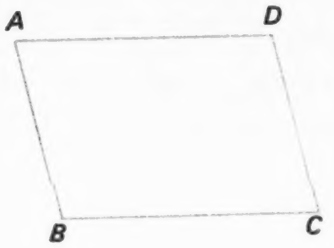


b. Using the same lengths as suggested above, construct another parallelogram. Make its shape different from the one you constructed above.

c. Again using the same lengths for sides, construct still another parallelogram. The shape of this parallelogram should be different from either of those already constructed.

d. How many parallelograms with different shapes can be constructed with sides that have the same lengths as segments PQ and QR?

13. a. Construct a parallelogram that looks like the one shown at the right. Label the figure $ABCD$.

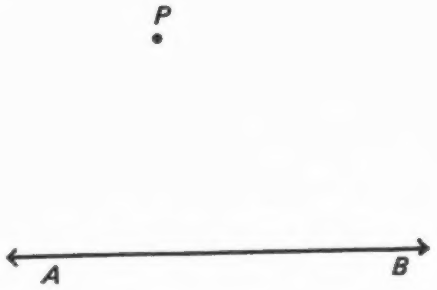


b. Through point A construct a perpendicular to the line that includes side BC. Label the point of intersection X. Segment AX is called an *altitude* of the parallelogram.

c. Suppose you constructed several parallelograms with sides that have the same lengths as the one pictured in the last exercise. Suppose also that in each case you constructed an altitude of the parallelogram. Would these altitudes all have the same length? Explain.

14. Make a copy of the diagram below.

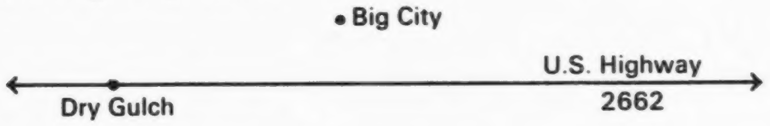
a. Through point P construct a line perpendicular to line AB. Label as M the point at which the perpendicular meets line AB.



b. Through point P construct a line perpendicular to line PM. Label this line PN.

c. Describe the position of line PN with respect to line AB.

15. Bob Plumb, the local civil engineer, wishes to build a road that will cross U.S. Highway 2662 at Dry Gulch. He wants this road to be perpendicular to the highway. He also wishes to build a second road, which will pass through Big City and be parallel to U.S. Highway 2662.



a. Make a copy of the map.

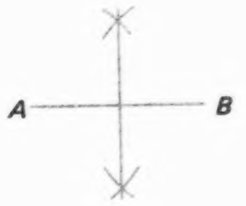
b. Use only a straightedge and a pair of compasses to construct lines that represent the new roads.

16. Without using a protractor, construct a parallelogram with an angle that has a measure of 45°.

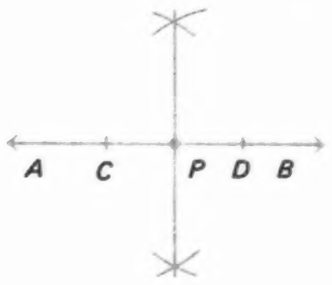
Summary—13

A summary of certain basic constructions

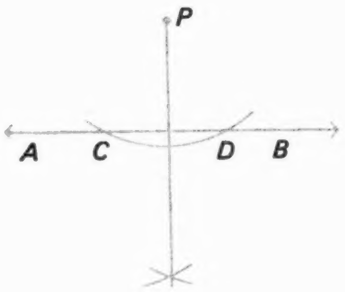
1. To construct the perpendicular bisector of a segment AB, make a rhombus in which segment AB is a diagonal. Then the endpoints of the other diagonal determine the perpendicular bisector of segment AB.



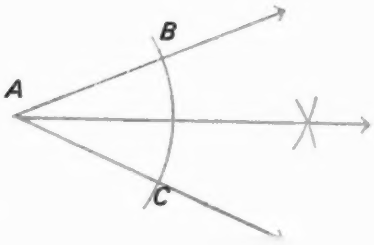
2. If P is a point in a given line AB, to construct a line through P that is perpendicular to line AB, first construct on line AB a segment CD that has P as its midpoint. Then construct a rhombus that has segment CD as a diagonal. The line determined by the endpoints of the other diagonal is perpendicular to line AB at P.



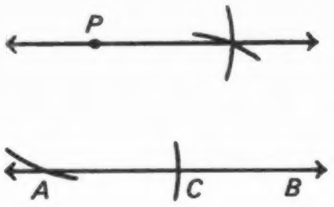
3. If P is a point that is not in line AB, to construct a line through P that is perpendicular to line AB, first use P as center to draw an arc of a circle that crosses line AB at two points C and D. Then construct a rhombus that has P, C, and D as vertices. The line that includes the diagonal having P as one endpoint is perpendicular to line AB.



4. To construct the bisector of a given angle A, begin by using point A as center and drawing an arc of a circle that crosses the sides of angle A. Label the points of intersection B and C. Construct a rhombus that has A, B, and C as vertices. The ray that has A as its endpoint and that includes the diagonal through A is the bisector of angle A.



5. If P is a point that is not in line AB, to construct a line through P that is parallel to line AB, first construct on line AB a segment AC that is congruent to segment PA. Then construct a rhombus that has points P, A, and C as vertices. The fourth vertex of the rhombus and the point P determine a line that is parallel to line AB.



General Summary

In this unit you have had an opportunity to study many important geometric properties. Among them are the following properties of triangles:

1. The sum of the measures of the angles of a triangle is 180°.

2. The sum of the lengths of any two sides of a triangle is greater than the length of the third side.
3. In a right triangle the square of the length of the longest side is equal to the sum of the squares of the lengths of the other two sides.

You have also had opportunity to study some properties of the rhombus. Recall that these properties were used in making constructions with a straightedge and a pair of compasses.

1. The opposite sides of a rhombus are parallel.
2. The diagonals of a rhombus are perpendicular.
3. The diagonals of a rhombus bisect each other.
4. Each diagonal of a rhombus bisects a pair of opposite angles of the rhombus.

Below is a list of geometric terms presented in this unit. The page on which a term is first used appears after the term.

VOCABULARY INDEX

Acute angle,30
Altitude of a parallelogram, 90
Altitude of a triangle, 82
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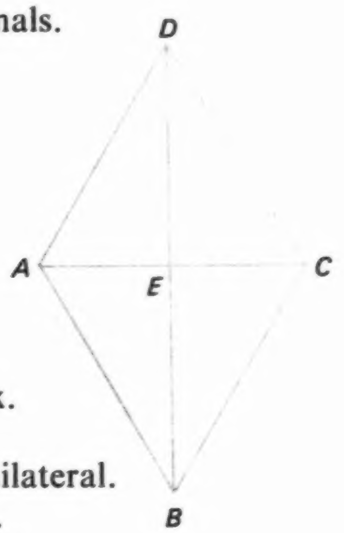
Review Exercises

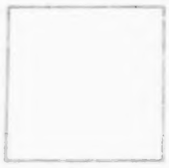
1. Consider the rhombus $ABCD$ and its diagonals.



- **a.** Name the diagonals of the rhombus.
- **b.** Name an angle that is bisected.
- **c.** Name two pairs of parallel sides.
- **d.** Name two right angles.
- **e.** Name a pair of segments that are perpendicular to each other.
- **f.** Name the midpoint of segment AC.
- **g.** Name four segments that are congruent.
- **h.** Name three angles that have C as a vertex.
- **i.** Name an angle that is an acute angle.
- **j.** Name an isosceles triangle that is not equilateral.
- **k.** Name a triangle that has an obtuse angle.
- **l.** Name an altitude of triangle ADC.

2. Which of the figures pictured at the right are—

(1) Quadrilaterials?
(2) Polygons?
(3) Trapezoids?
(4) Squares?
(5) Parallelograms?
(6) Rectangles?
(7) Rhombuses?
(8) Polyhedrons?
(9) Rectangular solids?
(10) Cubes?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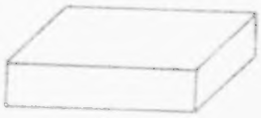
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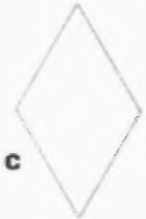
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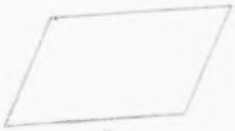
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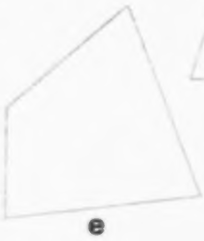
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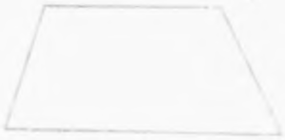
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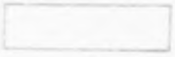
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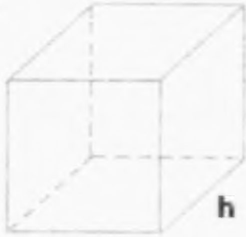

h

3. The figure pictured at the right is a polyhedron.

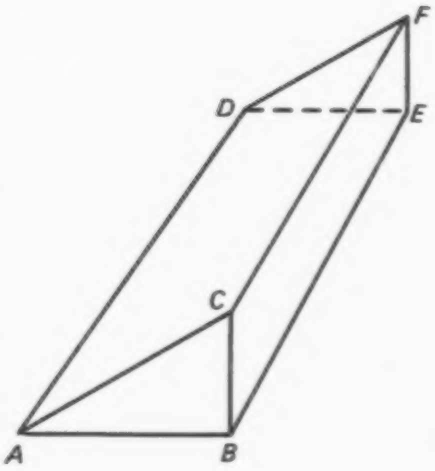
a. Name two faces that appear to be in parallel planes.

b. Name two segments that appear to be included in skew lines.

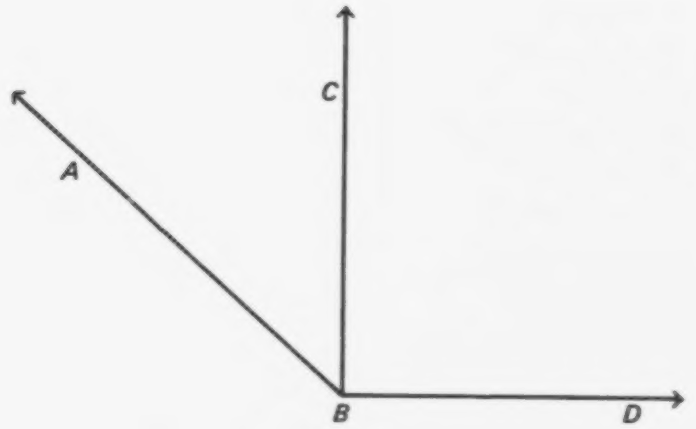
c. Name all the vertices of the polyhedron.

d. Name all the edges of the polyhedron.

e. Name all faces that are bounded by quadrilaterals.

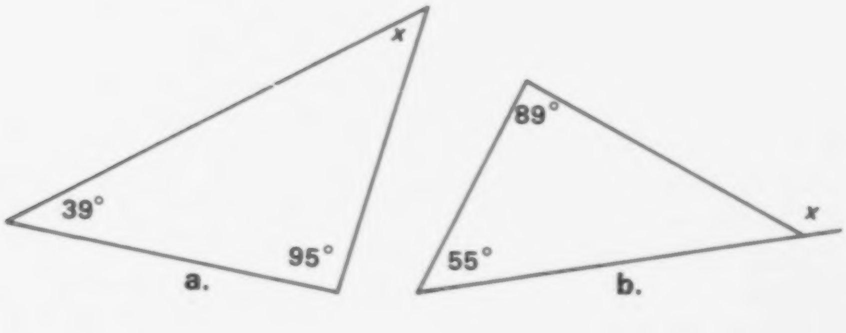


4. In the figure shown below, three rays have been drawn from a common endpoint *B*. Use your protractor to find the measure of each angle in the figure.



5. a. Draw an angle that has a measure of 129°.
   b. Draw an angle that has a measure of 85°.

6. Determine the measure of angle x in each figure shown below.

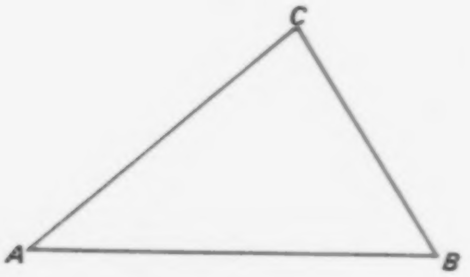


7. Construct a triangle having sides of 1 inch, 1½ inches, and 2 inches.

8. a. Draw a segment that is 2½ inches long.

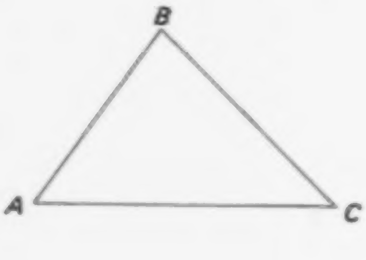
b. Construct the perpendicular bisector of the segment.
c. Construct a line through one endpoint of the segment that is perpendicular to the segment.
d. Two perpendiculars to the same line are ________________ to each other.

9. a. Make a copy of triangle *ABC*, and construct the three altitudes of the triangle.
   b. Make another copy of triangle *ABC*, and construct the three medians of the triangle.



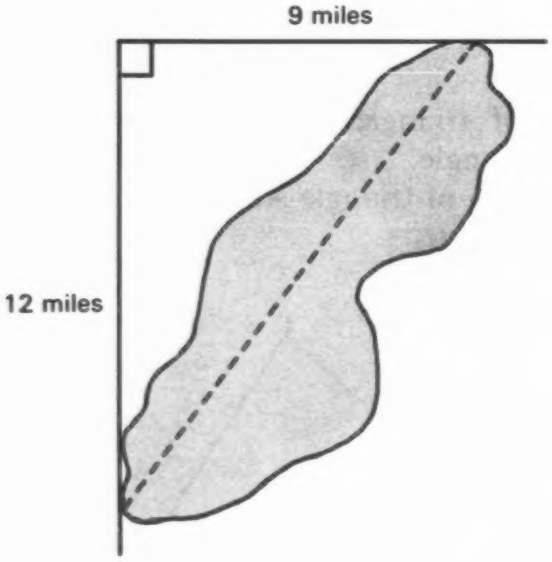
10. Draw an acute angle and bisect it, using a straightedge and a pair of compasses.

11. Draw a triangle on your paper like the one pictured at the right. Through point *B* construct a line parallel to the line that includes side *AC*.



12. Each set of three numbers listed below represents the measures of the sides of a triangle. If the triangles were drawn, which of them would be right triangles?
    a. 16, 20, 12
    b. 26, 10, 24
    c. 21, 28, 45

13. Bob Plumb wanted to find the length of Fishless Lake. The diagram below indicates the measurements that Bob made. Find the length of the lake.



5

Arrangements and Selections

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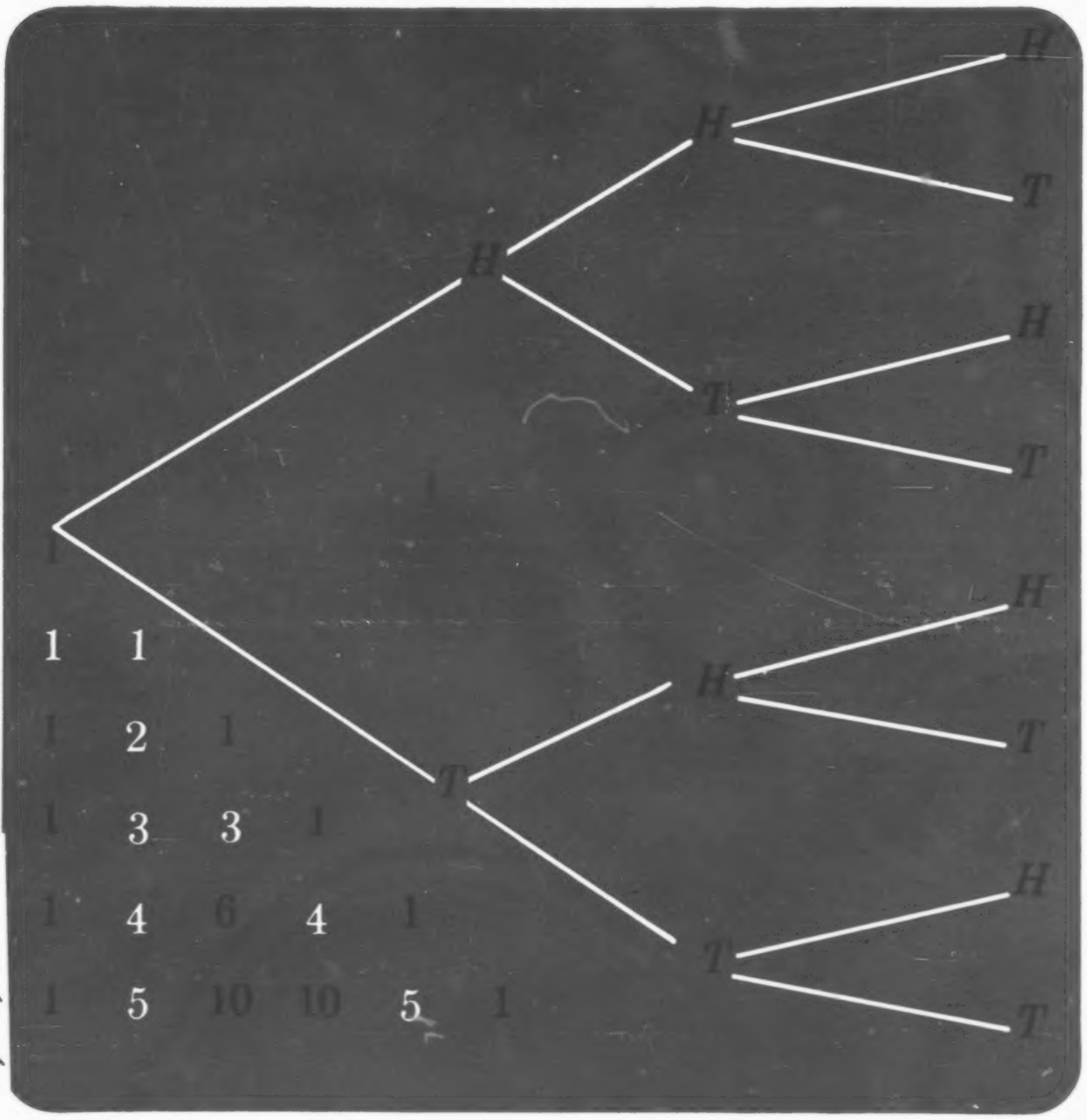
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UNIT FIVE OF

Experiences in Mathematical Discovery

Arrangements and Selections

NATIONAL COUNCIL OF
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Preface

"Experiences in Mathematical Discovery" is a series of ten self-contained units, each of which is designed for use by students of ninth-grade general mathematics. These booklets are the culmination of work undertaken as part of the General Mathematics Writing Project of the National Council of Teachers of Mathematics (NCTM).

The titles in the series are as follows:

Unit 1: *Formulas, Graphs, and Patterns*

Unit 2: *Properties of Operations with Numbers*

Unit 3: *Mathematical Sentences*

Unit 4: *Geometry*

Unit 5: *Arrangements and Selections*

Unit 6: *Mathematical Thinking*

Unit 7: *Rational Numbers*

Unit 8: *Ratios, Proportions, and Percent*

Unit 9: *Measurement*

Unit 10: *Positive and Negative Numbers*

This project is experimental. Teachers may use as many units as suit their purposes. Authors are encouraged to develop similar approaches to the topics treated here, and to other topics, since the aim of the NCTM in making these units available is to stimu-

late the development of special materials that can be effectively used with students of general mathematics.

Preliminary versions of the units were produced by a writing team that met at the University of Oregon during the summer of 1963. The units were subsequently tried out in ninth-grade general mathematics classes throughout the United States.

Oscar F. Schaaf, of the University of Oregon, was director of the 1963 summer writing team that produced the preliminary materials. The work of planning the content of the various units was undertaken by Thomas J. Hill, Oklahoma City Public Schools, Oklahoma City, Oklahoma; Paul S. Jorgensen, Carleton College, Northfield, Minnesota; Kenneth P. Kidd, University of Florida, Gainesville, Florida; and Max Peters, George W. Wingate High School, Brooklyn, New York.

The Advisory Committee for the General Mathematics Writing Project was composed of Emil J. Berger (*chairman*), St. Paul Public Schools, St. Paul, Minnesota; Irving Adler, North Bennington, Vermont; Stanley J. Bezuszka, S.J., Boston College, Chestnut Hill, Massachusetts; Eugene P. Smith, Wayne State University, Detroit, Michigan; and Max A. Sobel, Montclair State College, Upper Montclair, New Jersey.

The following people participated in the writing of either the preliminary or the revised versions of the various units:

Donald H. Firl, Rochester Public Schools, Rochester, Minnesota

Carroll E. Fogal, University of Florida, Gainesville, Florida

Edward F. Gottlieb, Madison High School, Portland, Oregon

Kenneth P. Kidd, University of Florida, Gainesville, Florida

Edna K. Lassar, Polytechnic Institute of Brooklyn, Brooklyn, New York

John F. LeBlanc, Racine Public Schools, Racine, Wisconsin

Scott D. McFadden, Spencer Butte Junior High School, Eugene, Oregon

Max Peters, George W. Wingate High School, Brooklyn, New York

Leonard Simon, Bureau of Curriculum Research, New York Public Schools, New York, New York

Irwin N. Sokol, Collinwood High School, Cleveland, Ohio

William H. Spragens, University of Louisville, Louisville, Kentucky

Mary E. Stine, Fairfax County Schools, Fairfax, Virginia

Arthur J. Wiebe, Pacific College, Fresno, California

John E. Yarnelle, Hanover College, Hanover, Indiana

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EMIL J. BERGER
Chairman, Advisory Committee
General Mathematics Writing Project

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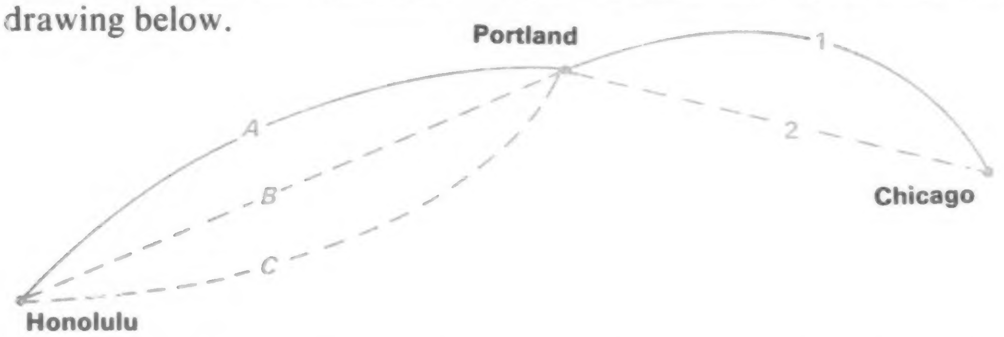
Experiences in Mathematical Discovery

Arrangements and Selections

1 A Trip to Hawaii

Phil, who lives in Chicago, is preparing to visit his uncle in Hawaii. He plans to travel to Portland by airplane and then take a ship to Honolulu. He can take either of two flights from Chicago to Portland and any one of three ships from Portland to Honolulu. These choices are shown in the drawing below.



One way in which Phil can make the trip from Chicago to Honolulu is to take Flight 2 and then sail on Ship *C*. We will agree to label this route 2*C*.

Class Discussion 1

1. Suppose Phil selects Flight 1 and Ship *B*. How will this route be labeled?
2. How will he travel if the route is labeled 2*A*?

3. If Phil decides to sail on Ship *C*, in how many ways can he make the trip from Chicago to Honolulu?

4. Label the routes that Phil can travel if he decides to sail on Ship *C*.

5. If Phil elects to take Flight 2, in how many ways can he make the trip? List the routes, using labels as in exercise 2.

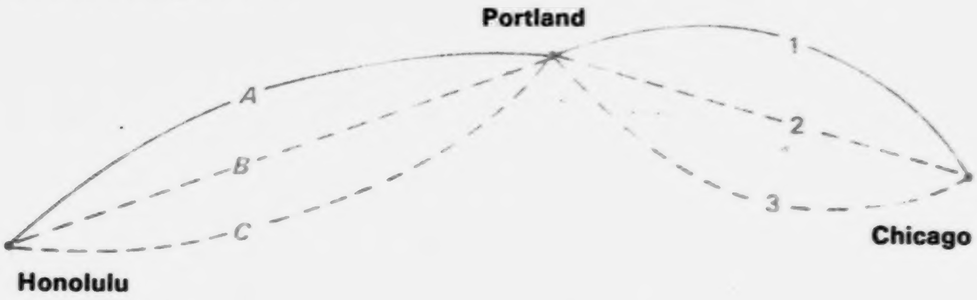
6. In how many ways can Phil make the trip if he selects Flight 1? List the routes.

Since Phil can travel to Portland by taking either Flight 1 or Flight 2 and in no other way, all possible routes that Phil can travel will have been included in the correct answers to exercises 5 and 6.

7. List all possible routes that are open to Phil.

8. What is the number of different routes?

Later, Phil learns that space is also available on Flight 3. With Flight 3 available to him, the choices that Phil can make are shown in the drawing below.



9. If Phil selects Flight 3, in how many ways can he make the trip from Chicago to Honolulu?

10. Label the routes that Phil can travel if he selects Flight 3.

11. Now make a new list of routes similar to the one that you made in exercise 7. Include the additional routes provided by the availability of Flight 3.

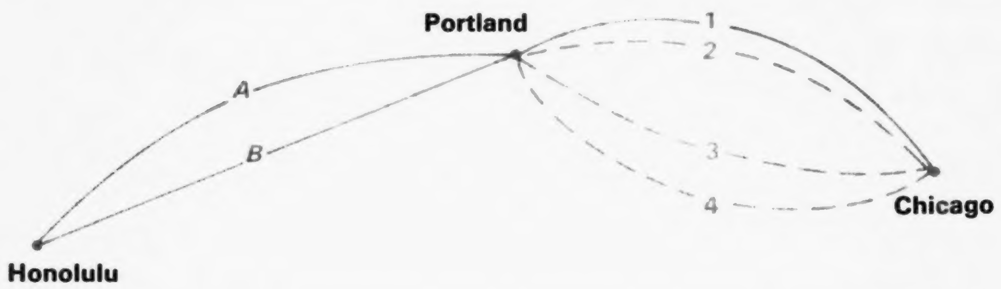
12. In how many different ways can Phil now make the trip from Chicago to Honolulu?

15

At this point it may be helpful to summarize the results we have obtained thus far. Entered in the table below are the results obtained in exercises 8 and 12. Make a copy of this table and keep it for use later. Be sure to include the empty spaces that are not yet filled in. Soon you will be asked to enter other results in your table.

Number of flights	Number of ships	Total number of routes
2	3	6
3	3	9

When planning his trip Phil finds out that if he delays his trip three weeks, he can have a choice of four flights from Chicago to Portland and a choice of two ships from Portland to Honolulu. These choices are represented in the drawing below.



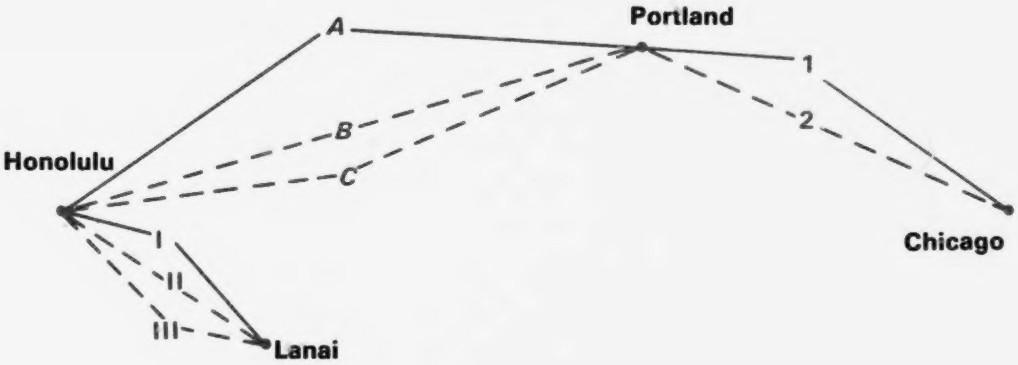
13. Label the routes that Phil can travel if he has a choice of four flights and a choice of two ships.

14. In how many ways can Phil make the trip if he waits three weeks? Enter this information in the table you were asked to make.

15. If there is no room on Ship *B*, leaving only Ship *A* and four flights, in how many ways can Phil make the trip? Enter this information in your table.

16. Study your table carefully. Can you discover any pattern or relationship? If you can, state it in your own words.

Phil's uncle lives on the island of Lanai, not far from Honolulu. Helicopters fly passengers from Honolulu to Lanai three times each day. If we go back to the first set of choices Phil had—that is, two flights from Chicago to Portland and three ships from Portland to Honolulu—the problem of selection would look like this:



17. We already know there are six ways to travel from Chicago to Honolulu. If only Flight I were available from Honolulu to Lanai, how many ways would there be of getting from Chicago to Lanai?

18. If only Flight II were available, how many ways would there be of getting from Chicago to Lanai?

19. If both Flight I and Flight II were available, how many ways would there be of getting from Chicago to Lanai?

20. How many ways would there be of getting from Chicago to Lanai if all three flights were available from Honolulu to Lanai?

21. If m is the number of flights out of Chicago, n the number of ships sailing from Portland to Honolulu, and p the number of helicopter flights that are available, how would you determine the number of routes for the complete trip from Chicago to Lanai?

Exercises—1

1. Jane has arranged to stop at Susan's house after school. Jane can get to Susan's house on any one of four buses. Later, she will have a choice of five rides to get home. In how many different ways can Jane go home from school?

2. Tom is selecting a seat in the auditorium. There are 14 rows and 20 seats in each row. First Tom selects a row; then he selects a seat in the row of his choice. In how many ways can Tom select a seat in the auditorium?

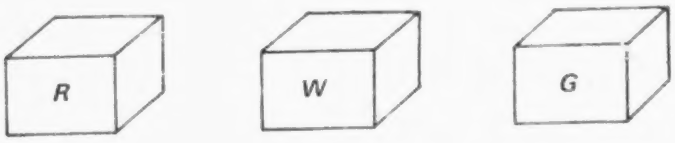
3. Robert has two suits and three ties in his wardrobe. Let *S* represent one of the suits, and let *T* represent the other. Also, let the numerals 1, 2, and 3 respectively represent the three different ties. Listing a suit and a tie in that order, indicate all possible suit-and-tie combinations. (*Hint:* Use the symbol *S*1 to indicate the combination composed of the suit labeled *S* and the tie labeled 1.)

4. How many different combinations did you find in exercise 3?

5. In baseball, a battery consists of the pitcher and the catcher. If a team has seven pitchers and three catchers, how many different batteries can be formed?

6. If, in addition to the two suits and three ties mentioned in exercise 3, Robert has five shirts in his wardrobe, how many different suit-tie-and-shirt combinations are possible?

7. A box contains three green blocks, three red blocks, and three white blocks. Let *G* represent a green block, *R* a red block, and *W* a white block. Jim is to draw three blocks out of the box at random and to arrange them in a row. One possible arrangement would be *RWG*; another would be *RRR*. List all possible arrangements. It will help to list first all possible arrangements that have *R* as the first letter on the left, then those with *W* on the left, and finally those with *G* on the left.



8. If a coin is tossed, the possible outcomes are H (heads) and T (tails). Suppose the coin is tossed twice and the outcome of each toss is recorded. Name all possible outcomes of two tosses. (*Hint:* One such outcome would be HT, which means heads on the first toss and tails on the second.)

9. A coin is tossed three times. The result of each toss is recorded. One possible result of three tosses is TTH. Name all possible results of three tosses.

10. A coin is tossed four times, and the results are recorded as in exercises 8 and 9. How many different outcomes of four tosses can occur?

11. Enter the information from exercises 8, 9, and 10 in a table similar to the one shown below. Then fill in the spaces that remain. State a rule for determining the number of different results that can occur if you know how often the coin is tossed.

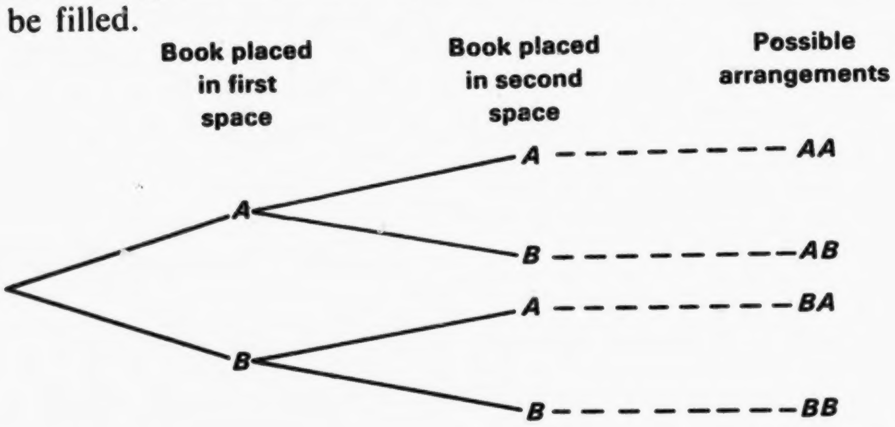
Number of tosses	**Number of possible results**
1	
2	
3	
4	
5	
6	

2 A Tree Grows in Mathematics

We can now summarize what we have discovered about two events. Remember Phil's first travel problem. He had two choices of flights from Chicago to Portland and three choices of ships from Portland to Honolulu. We found that he could make the trip in 2 x 3 = 6 different ways. If a first thing can be done in m different ways and, after it is done, a second thing can be done in n different ways, then the two things together can be done in $m \cdot n$ different ways.

There are other ways in which this rule can be illustrated, as may be seen in the following problem. Suppose we have several identical copies of Book A and several identical copies of Book B. There are two spaces on a shelf to be filled. In how many ways can you fill the shelf with books so that the arrangements look different?

The drawing below, called a *tree diagram*, shows a method of determining the different ways in which two spaces on the shelf can be filled.

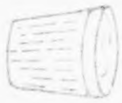


We can also indicate the number of different ways in which the two spaces can be filled by using a box diagram as shown below.

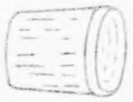
Number of ways the first space can be filled	Number of ways the second space can be filled	Number of possible arrangements
2 ×	2 =	4

Using our rule for two events, we see that $2 \cdot 2 = 4$. This means that the two spaces can be filled in four ways that look different. The tree diagram shows the different arrangements.

Now let us look at another problem. Below is a picture of a cap for a tube of toothpaste.



If the cap is tossed in the air, it will land and come to rest in one of three positions.



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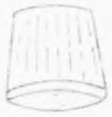

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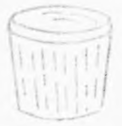


T

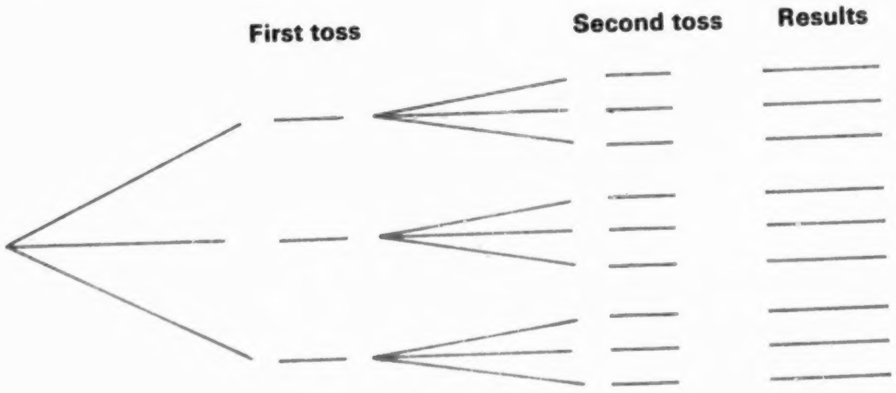
Let S represent the case in which the cap comes to rest on its side, B the case in which the cap comes to rest on its bottom, and T the case in which the cap comes to rest on its top. Suppose that we have two caps and that we toss each in turn. We can indicate one of the possible outcomes by writing BT. These two letters tell us that the first cap comes to rest on its bottom and that the second cap comes to rest on its top.

Class Discussion **2a**

1. In how many ways can the first cap come to rest?
2. In how many ways can the second cap come to rest?
3. Enter your answers to questions 1 and 2 in the box diagram below. Then compute the number of possible outcomes of tossing two caps.

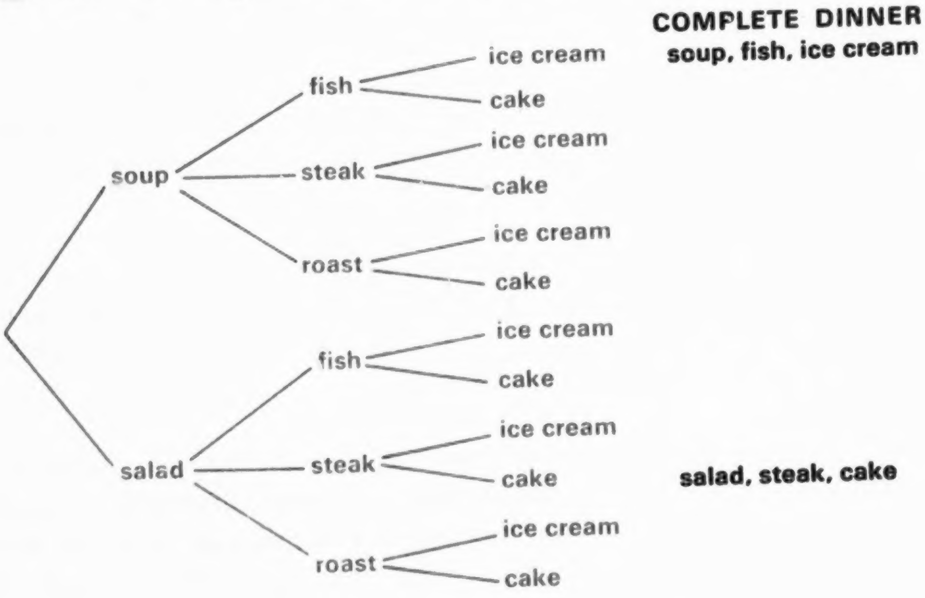
Number of ways first cap can come to rest	Number of ways second cap can come to rest	Number of possible outcomes
	×	=

4. Display the results with a tree diagram.


First toss
Second toss
Results

Class Discussion 2b

1. The menu in a certain restaurant offers a choice of soup or salad for an appetizer; a choice of fish, steak, or pork roast for the entrée; and a choice of ice cream or cake for the dessert. A complete dinner consists of one choice from each group. In how many different ways can you choose a complete dinner? Use the tree diagram below to help you answer the question.


COMPLETE DINNER
soup, fish, ice cream
soup
fish
ice cream
cake
steak
ice cream
cake
roast
ice cream
cake
salad
fish
ice cream
cake
steak
ice cream
cake
salad, steak, cake
roast
ice cream
cake

A box diagram like the one below can be used to organize the information in the tree diagram.

Number of appetizers	Number of entrées	Number of desserts	Total number of complete dinners
2 ×	3 ×	2 =	

2. We already know that if a first thing can be done in m ways and a second thing in n ways, then the two things together can be done in $m \cdot n$ ways. Now, if a first thing can be done in m ways, a second thing in n ways, and a third thing in p ways, in how many ways can the three things be done?
3. If a fourth thing can be done in q ways, in how many ways can all four things be done?

From this discussion it would appear that the number of different dinners that can be selected from the restaurant menu is $2 \cdot 3 \cdot 2 = 12$. Do you agree? Would you also agree that the answer to exercise 3 is $m \cdot n \cdot p \cdot q$?

Exercises—2

1. In a seven-story building the rooms are labeled with a numeral followed by a letter of the alphabet. If the numeral indicates the floor and the letter the room, how many rooms can be labeled differently?

2. A certain school uses only two-digit numerals on its football uniforms. How many uniforms can be labeled differently if only the digits 2, 3, 4, 5, 6, 7, and 8 are used? Use a box diagram to help you answer the question.

3. At a certain drive-in, hamburgers may be ordered as follows: the bun may be toasted or untoasted; catsup or mustard or both or neither may be spread on the bun; the hamburger may be served plain or with relish, pickles, or onions, but with no more than

one of these; and the hamburger may be prepared with or without lettuce. How many different kinds of hamburgers can be ordered if a choice is made in each case?

4. John pitches for his school's baseball team. In practice, he frequently makes a record of his accuracy. He throws three times and then records the results, writing *S* for a strike and *B* for a ball. *BSS* means a ball followed by two strikes. Draw a tree diagram illustrating all possible results. How many different results are possible?

5. A certain state makes automobile license plates by stamping two letters followed by two digits in a rectangular-shaped piece of metal. All letters except O, Q, and I are used. How many different plates can be manufactured? (Remember that the plate "AA 00" is to be included.)

6. Another state uses two letters followed by three digits. If all the letters of the alphabet are used, how many different plates can be manufactured?

7. A third state is considering changing from the system described in exercise 6 to one in which three letters are followed by two digits. Would such a change increase the number of plates that can be manufactured?

8. What would be the difference in the number of plates that can be manufactured under the two systems described in exercises 6 and 7?

9. How many different telephone numbers are possible if the numeral for each number consists of seven digits?

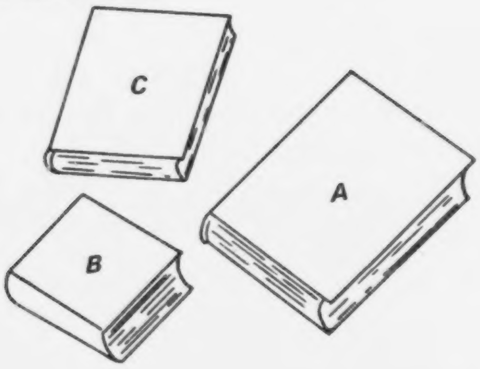
10. How many different telephone numbers are possible if the first digit is not zero?

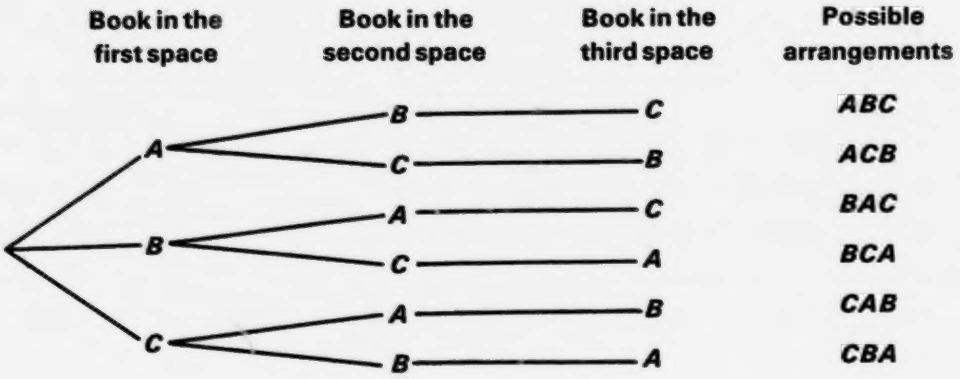
11. Find out what identification scheme your state uses in manufacturing automobile license plates. How many different plates can be manufactured using this scheme?

3 Arranging a Book Display

Mary works in the library. Today she is to arrange three new books in a display. To find the best arrangement, she decides to try them all. She labels the books *A*, *B*, and *C*, and labels the spaces into which they will go the first, second, and third spaces.

In how many different ways can the books be arranged? Let's use a tree diagram to find out.




Book in the first space
Book in the second space
Book in the third space
Possible arrangements
A
B
C
B
C
A
C
A
B
C
A
B
C
B
C
A
B
A
ABC
ACB
BAC
BCA
CAB
CBA

Class Discussion 3

1. In how many ways can the first space be filled?
2. If Book *A* is placed in the first space, in how many ways can the second space be filled?

3. If Book B is placed in the first space, in how many ways can the second space be filled?

4. In general, after the first space is filled, in how many ways can the second space be filled?

5. After the first space has been filled with Book A and the second space with Book B, in how many ways can the third space be filled?

6. In general, after the first and second spaces have been filled, in how many ways can the third space be filled?

7. Use a box diagram of the type shown below to record all information.

Number of ways first space can be filled	Number of ways second space can be filled if first space is filled	Number of ways third space can be filled if first and second spaces are filled	Number of ways books can be arranged
×	×	=	

8. How many branches are there in the tree diagram from the beginning point to the list of possible choices of books for the first space?

9. How does this compare with the first entry in the box diagram?

10. How many branches are there from any given book in the first space to the list of possible choices of books for the second space?

11. How does this compare with the second entry in the box diagram?

12. How many branches are there from any given book in the second space to the list of possible choices of books for the third space? Does this agree with the third entry in the box diagram?

Exercises—3

1. If the librarian had asked Mary to arrange four books in the display, how many arrangements would have been possible? Draw a tree diagram to illustrate your answer.

2. Prepare a box diagram for the information given by your tree diagram in exercise 1.

3. Don has a model train with the following equipment in addition to the engine and caboose: tank car, flatcar, boxcar, refrigerator car, and stockcar. In how many ways can he arrange the five cars between the engine and the caboose? Prepare a box diagram.

4. Study the number sequences in the box diagrams used in exercise 7 of Class Discussion 3 and in exercises 2 and 3 above. Do you see any pattern?

5. If you detected a pattern in exercise 4, predict what the pattern would be if Don had six cars to arrange.

6. Prepare a box diagram for the case in which Don has six cars to arrange.

7. How many arrangements are possible with six cars?

8. If this pattern holds, what numbers would you multiply to get the number of arrangements for eight cars?

9. Ten people are lined up in a row to have a picture taken. The photographer is very particular, and he keeps shifting people around. In how many different ways can he arrange the ten people if they must stand in a row?

4 The Team Picture

The championship basketball team of Rosedale High School is posing for its official photograph. The nine players and the coach are being arranged and rearranged by the photographer. Determined to get the best picture, he is trying out all possible arrangements. He makes a new arrangement every six seconds. This is at the rate of 600 per hour, or 4,800 per eight-hour working day. If he works every day of the year, it will take two years and 26 days to photograph all possible arrangements.

Previously you found that four things can be arranged in $4 \cdot 3 \cdot 2 \cdot 1$, or 24, ways. Five things can be arranged in $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, or 120, ways.

Mathematicians use an exclamation mark (!) to indicate the type of product we have been considering. Thus $4 \cdot 3 \cdot 2 \cdot 1$ is symbolized by writing 4!. This is read "four factorial." The order of the factors does not matter; that is, $4 \cdot 3 \cdot 2 \cdot 1 = 1 \cdot 2 \cdot 3 \cdot 4 = 4!$. In the case of the basketball team there were 10! possible arrangements. The number of possible arrangements in a line of n things is $n!$. That is, $n!$ means the product of the first n counting numbers.

Class Discussion 4

1. Use factorial notation to indicate the number of ways in which 12 different books can be arranged in a row on a shelf.

2. In how many ways can a freight train of 85 cars be coupled together? Use factorial notation in writing your answer.

3. A parade with 37 floats is being arranged. Use factorial notation to indicate the number of different ways in which 37 floats can be arranged.

4. 2! is the product of what two numbers?

5. Find the product of 2! and 3!.

6. 4! multiplied by what number equals 5!?

7. What number equals 5! ÷ 3!?

Exercises—4

1. Use factorial notation to indicate in how many ways seven people can line up to buy tickets to a baseball game.

2. Compute the number represented by the symbol 6!.

3. Do the same for 7!.

4. What number is represented by the symbol 8!?

5. In how many ways can a baseball manager arrange the batting order of nine players? First give the answer in factorial notation; then calculate the actual number of ways.

5 Coach Johnson Has a Problem

There are five excellent candidates for the outfield positions on the Rosedale High School varsity baseball squad. Tom, Doug, Bill, John, and Mike are all hitting above .300, and they field the ball equally well. Three positions are open—right field, center field and left field. In how many different ways can Coach Johnson choose his outfield?

Class Discussion 5

1. Construct a tree diagram that illustrates the choices Coach Johnson can make. Assume that the right-field position is to be filled first, the center-field position second, and the left-field position last.

2. In how many ways can Coach Johnson choose the right fielder?
3. After this choice is made, in how many ways can he choose the center fielder?
4. With these two positions filled, in how many ways can he choose the left fielder?
5. Construct a box diagram and enter your answers from exercises 2, 3, and 4.
6. In how many ways can the outfield be chosen?
7. Had there been six candidates instead of five, what would the number of choices have been?
8. If there are seven candidates for the four infield positions (first base, second base, third base, and shortstop), in how many ways can the infield be chosen?
9. If there are eight candidates for the four infield positions, in how many ways can the infield be chosen?

Exercises—5

1. In a club of 12 members an election is held to choose a president, a secretary, and a treasurer. In how many different ways can the three positions be filled?

2. A chorus concert consists of 13 selections. The first part of the concert is composed of 4 of these selections. In how many ways can the first part of the concert be planned?

3. Ann, Betty, Sue, and Janet are prepared to give their speeches in English class. In how many ways can the teacher arrange the order of presentations?

4. How many four-digit numerals can be formed with the digits 1, 2, 3, 4, and 5 if no digit is used more than once in a numeral?

5. Three girls are looking at four sweaters, each of which is a different color. In how many different ways can the sweaters be distributed among the girls if each girl is to get one sweater?

6. Three boys enter a room in which ten chairs are arranged in a row. In how many ways can the boys seat themselves? (*Hint:* Begin by determining the number of ways in which the first boy can choose a chair.)
7. A two-flag signal is arranged on a flagpole by placing one flag at the top and a second flag just below it. The order is important. If the positions of the flags are changed, it means that a different signal is being given. How many signals can be given if five different flags are available?
8. If the number of flags in exercise 8 is increased to seven, how many different two-flag signals can be given?
9. In how many ways can three students be seated in a classroom containing 30 chairs?
10. In the game of Scrabble a tile is labeled by one of the 26 letters of the alphabet, or it is blank. Suppose that a set of tiles contains exactly one tile of each kind. In how many different ways can three tiles be drawn and placed in a row?
11. A relay team is composed of four runners, A, B, C, and D. In how many different orders can the runners be arranged for a relay race?
12. Robert can travel to work with any one of three friends. He can return home with any one of five workers at the plant. In how many different ways can Robert choose to travel the round trip?
13. In how many ways can six boys and six girls pair off to skate as couples? (Assume that a couple consists of one boy and one girl.)

Summary—5

You may have observed that the problems we have considered thus far in this unit have been solved by the same general method. In each case the *fundamental principle of ar-*

rangements was used. (Mathematicians refer to this principle as the *principle of counting*.) According to this principle, if a first thing can be done in m ways, and if after that a second thing can be done in n ways, and if after that a third thing can be done in p ways, then the number of ways in which the three things together can be done is $m \cdot n \cdot p$. This principle, as you have seen, generalizes to cases in which the number of consecutive actions is greater than three.

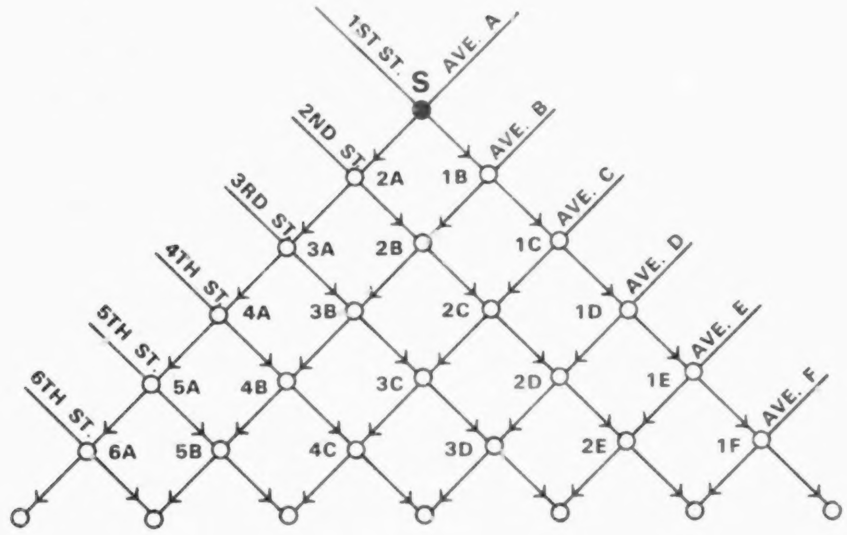
Below is a table that summarizes some of the problems we have considered thus far.

APPLICATION OF THE FUNDAMENTAL PRINCIPLE

		First thing done in m ways	Second thing done in n ways	Third thing done in p ways	Total number of ways = $m \cdot n \cdot p$
PROBLEM	A Trip to Hawaii	2 flights to Portland	3 ships to Honolulu	3 helicopters to Lanai	$2 \cdot 3 \cdot 3 = 18$
	Choosing a Dinner	2 choices of appetizer	3 choices of entrée	2 choices of dessert	$2 \cdot 3 \cdot 2 = 12$
	Arranging a Book Display	3 ways to fill first position	2 ways to fill second position	1 way to fill third position	$3 \cdot 2 \cdot 1 = 6$
	Coach Johnson Has a Problem	5 ways to choose right fielder	4 ways to choose center fielder	3 ways to choose left fielder	$5 \cdot 4 \cdot 3 = 60$

6 Pascal City

In Pascal City the city council has passed a regulation making all streets and avenues in the central part of the city one-way. The council has also ruled that the direction of travel on all avenues shall be the same, and that the direction of travel on all streets shall be the same. A map of the central part of the city is shown below.


1ST ST.
AVE. A
S
2ND ST.
AVE. B
3RD ST.
AVE. C
4TH ST.
AVE. D
5TH ST.
AVE. E
6TH ST.
AVE. F
2A
1B
3A
2B
1C
4A
3B
2C
1D
5A
4B
3C
2D
1E
6A
5B
4C
3D
2E
1F

Class Discussion 6

In the exercises that follow, always start your trips at S and travel in the direction of the arrows. To do the exercises, you will need to have a copy of the above map. Make all entries requested on your own map.

1. In how many ways could you travel from S to the intersection of 2nd and A (labeled 2A)? Write your answer in the circle at 2A.
2. In how many ways could you travel from S to 2nd and B? Enter your answer in the circle at 2B.
3. In how many ways could you travel from S to intersection 3A? From S to intersection 1C? Enter each answer in the circle that appears at the intersection named.
4. What are the last intersections you could pass through before you come to 2C?
5. In how many ways could you travel to get to 2B and then to 2C?
6. In how many ways could you travel to get to 1C and then to 2C?

7. Are there any other legal ways of getting to 2C?

8. From your answers to questions 5, 6, and 7, can you tell how many routes there are to 2C? If so, how many?

9. How many routes are there to 3B?

10. Through what "last" intersections can you pass in order to reach 3C?

11. How many ways are there of getting from S to 3C?

12. How many ways are there of getting from S to 1C? 1D? 1E?

13. The intersections S, 1B, 1C, 1D, 1E, 1F, 2A, 3A, 4A, 5A, and 6A are boundary intersections. Select any intersection that is not a boundary intersection. From how many last intersections can you leave and arrive at the intersection you selected?

14. How many routes are there to each boundary intersection listed in exercise 13?

15. Complete the entries in the circles at all intersections.

After making all entries on your map as directed, study the resulting array of numbers to see if you can discover a pattern. Without the arrows and lines, and except for the fact that we have left out the last row of numbers, the array should look like this:

S

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

If we replace the S in the top position by 1, the resulting array of numbers is said to be in the form of Pascal's triangle. As you look at the array, you may begin to notice certain things. Pick out any two numbers that are next to each other in a given row—say, for example, 4 and 6. Now look at the number that is directly below the space between them.

4 6

10

Is there a relation between the two numbers 4 and 6 and the number 10? Does this same relation exist in other parts of the triangle? There are many interesting things about Pascal's triangle which we shall be looking into soon.

The triangle and our imaginary city are both named in honor of Blaise Pascal, a French mathematician born in 1623. At the age of nineteen he invented and built the first officially recorded machine for adding and subtracting. The speedometer on a car, the cyclometer on a bicycle, and the fare register on a streetcar are all adaptations of Pascal's machine. Pascal, of course, did much more than invent a machine with which to add and subtract. He also made important contributions to geometry and probability theory.

Exercises—6

1. Use intersection entries to build a triangular array of numbers that has the form of Pascal's triangle. Be sure to keep track of the street numbers and the avenue letters. Replace S by 1 and continue building the triangle until you get a horizontal row that contains the number 252.

2. This exercise is concerned with certain number patterns. By studying the pattern suggested by the numbers that are given, you should be able to decide what numbers come

next. The first set should be easy. Those that follow may be a little harder. In each case fill in the blank spaces in such a way that the pattern indicated by the given numbers is continued.

a. 1 2 3 4 ___ ___ ___ ___

b. 1 3 6 10 ___ ___ ___ ___

c. 1 4 10 20 ___ ___ ___ ___

d. 1 5 15 35 ___ ___ ___ ___

3. Look at your results in exercise 2a. Then look down Second Street of your map of Pascal City. What do you observe? How is the pattern formed?

4. Study your results in exercise 2b. Now look down Third Street or Avenue C of your map. What do you observe? How is the pattern formed?

5. Compare your results in exercise 2c with Fourth Street. What do you observe?

6. On which street would you look for the pattern in exercise 2d?

7. What would be the easy way to complete the following pattern?

1 6 21 56 ___ ___ ___ ___

8. How many different routes are there for arriving at the intersection in which you have written 252?

9. Look at your map and locate one of the intersections that contain the number 210 and the intersection that contains the number 252. Starting at S, how many ways are there of arriving at the intersection into which both feed?

7 Pascal's Triangle

Class Discussion 7

1. Let's have another look at Pascal's triangle. This time find the sum of the numbers in each horizontal row.

1 = 1
1 + 1 = 2
1 + 2 + 1 = ___
1 + 3 + 3 + 1 = ___
1 + 4 + 6 + 4 + 1 = ___
1 + 5 + 10 + 10 + 5 + 1 = ___
1 + 6 + 15 + 20 + 15 + 6 + 1 = ___

2. Enter the sums for the rows in the blanks below.

1 2 ___ ___ ___ ___ ___

3. Do you detect a pattern in these sums? Suppose there were a few more rows in the triangle above. What would the sum of the numbers in each of these new rows be?

Exercises—7

In this set of exercises we shall investigate some other ideas based on Pascal's triangle. To discover what these are, you will need to make use of the map of Pascal City in Section 6, or your own more complete map. Be sure to place a "1" in the circle at intersection S before beginning these exercises.

1. Add the numbers in the intersections along First Street as you walk from S to 1E. Include the 1 in intersection S. What sum do you obtain?

   Imagine that you are standing in intersection 1E. Look down Avenue E in the direction of the arrow to the next intersection.

What number is in that intersection? How does this number compare with your sum?

2. This time stroll down Second Street from 2A to 2D. Add the intersection numbers. What sum do you obtain?

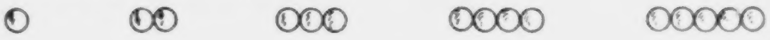
From 2D look down Avenue D to the next intersection. What number is in that intersection?

3. This time take a walk down Avenue C from 1C to 4C, and again add the numbers in the intersections that you encounter. What sum do you obtain?

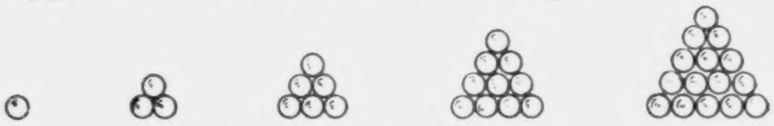
Since you are walking down the avenue, glance down the street from 4C and read the number in the next intersection. How does this number compare with your sum?

4. Repeat the procedure described in exercises 1, 2, and 3 with any street or avenue. Remember, if you walk along a street, look down an avenue, and vice versa.

5. The numbers along Second Street are 1, 2, 3, 4, 5, and so on. These numbers are sometimes called the *counting numbers*. If we represent each of these numbers by marbles placed in a row, the rows will look like this:

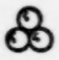
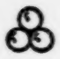
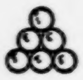
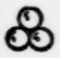
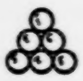
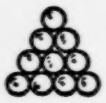
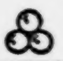
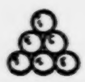
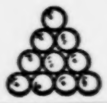
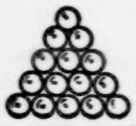


Now suppose we use marbles to form triangles:



Count the number of marbles in each triangle, starting at the left. What sequence of numbers do you get? The numbers in this sequence are called triangular numbers. Does the pattern in this sequence appear somewhere in Pascal's triangle? How many marbles would the sixth triangle contain?

6. Pyramidal numbers can be found by building triangular pyramids with marbles. The layers in each pyramid will be a set of triangles like those pictured in exercise 5. The display below indicates how many marbles there will be in the various layers of five different pyramids.

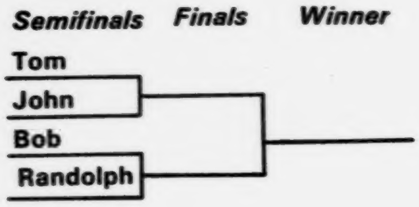
	Pyramid *A*	**Pyramid *B***	**Pyramid *C***	**Pyramid *D***	**Pyramid *E***
Fifth Layer	**none**	**none**	**none**	**none**	
Fourth Layer	**none**	**none**	**none**		
Third Layer	**none**	**none**			
Second Layer	**none**				
First Layer					

Find the number of marbles in each pyramid. What sequence of numbers do you get for pyramids *A*, *B*, *C*, *D*, and *E*? Do you detect a pattern in the sequence? Does this pattern appear in Pascal's triangle?

7. In exercise 5 the counting numbers are represented by rows of marbles. Thus, the counting numbers can be considered one-dimensional. Triangular numbers may be thought of as two-dimensional, and pyramidal numbers as three-dimensional. Where in Pascal's triangle is there a suggestion for a four-dimensional figure?

8. You are familiar with tournament schedules. These resemble tree diagrams in reverse. Below is a diagram of such a schedule for the semifinals and finals of a table-tennis tournament.

Complete the table to show that Bob defeated Randolph, John defeated Tom, and John defeated Bob.



9. How many places are there in the finals? The semifinals? The quarterfinals?

10. The number of entries in a tournament does not always fit the tree. If there were seven entries, play would start in the quarterfinals; however, one spot would be open. The person paired with this open spot would be said to draw a "bye." He would advance to the next round without playing. In a tournament it is customary to draw (or assign) all byes in the first round. How many players would draw byes if there were six entries in a tournament? How many players would draw byes if there were five entries?

11. Sixteen players can play in the round just before the quarterfinals. How many can play in the round before that?

12. If thirteen players enter a tournament, how many players will draw byes? How many rounds will be required?

13. Fifty-four golfers enter a match-play tournament. If the golfers are paired in the same way as were contestants in the table-tennis tournament, how many byes would there be in the first round?

14. How many rounds are needed for the golf tournament described in exercise 13?

15. Ten basketball teams enter a tournament and are scheduled in the manner described in exercise 8. How many byes need to be assigned in the first round?

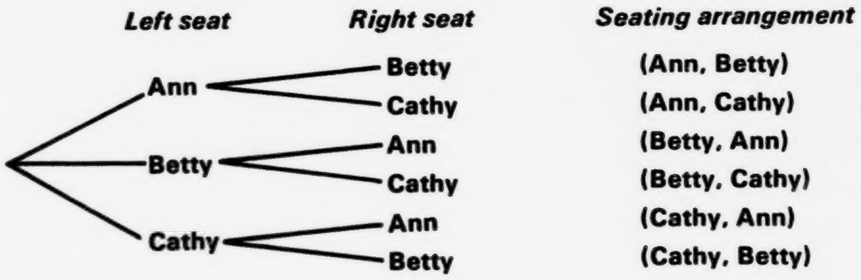
16. Arrange the following basketball tournament on a tree diagram: In the first round Antelope Valley defeated Kingston, Carmel drew a bye, Lakeville defeated Sheldon, and Summit defeated Riverdale. In the second round Summit defeated Carmel, and Lakeville defeated Antelope Valley. Lakeville defeated Summit in the finals.

8 Seats at the Game

Three girls—Ann, Betty, and Cathy—are late in arriving at the finals of the basketball tournament. There are only two seats left. This presents a problem. Which two girls are to have the seats? In how many ways can the choice or selection be made? This problem is somewhat different from those we have worked with previously. Let's see if we can understand the difference.

Class Discussion 8a

1. The girls could decide that Ann and Betty should have the seats. Describe other selections that could be made.
2. How many different selections are possible?
3. In counting the number of different selections, you should have counted (Ann, Cathy) as one selection. Would you count (Cathy, Ann) as a different selection? The answer is "no" because we are concerned only with the problem of which two girls get the seats and not with the order in which they sit. There are three possible selections.
4. But suppose now that we look at this problem in the same way that we looked at some problems in a previous set of exercises and ask ourselves, How many different seating arrangements are possible for the three girls? When we consider arrangements, we are concerned with who sits on the right and who sits on the left. Using a tree diagram as before, we can show all possible ways of filling the left and right seats.


Left seat
Right seat
Seating arrangement
Ann
Betty
Cathy
Betty
Cathy
Ann
Cathy
Ann
Betty
(Ann, Betty)
(Ann, Cathy)
(Betty, Ann)
(Betty, Cathy)
(Cathy, Ann)
(Cathy, Betty)

How many different seating arrangements are there?

5. The number of possible arrangements is how many times the number of possible selections we counted earlier? Can you explain why this is so?

Let us note the difference between the terms *arrangements* and *selections*.

In counting the number of arrangements we say that the arrangement

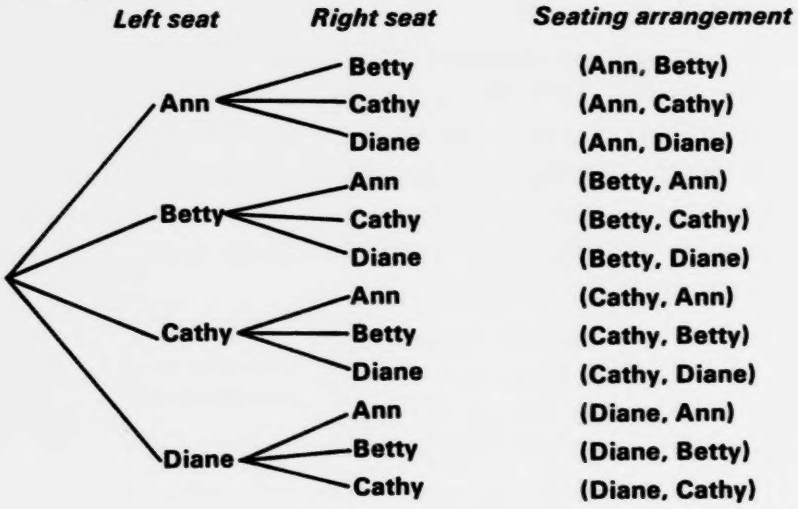
(Ann, Betty),

which indicates that Ann has the left seat and Betty has the right seat, is different from the arrangement

(Betty, Ann).

But in a selection we are only concerned with which two girls are chosen. Thus the two different arrangements shown above are counted as only one selection. In a problem involving the number of different selections, the order, or placement, *does not count*. When we are asked to give the number of different arrangements, however, the order *does* count. This idea will become clearer as we look at other examples.

6. Suppose that there are four girls—Ann, Betty, Cathy, and Diane—and that, as before, only two seats are available. To determine how many different seating arrangements are possible, we again make use of a tree diagram. This time, however, our tree is a bit more complicated.


Left seat
Right seat
Seating arrangement
Ann
Betty
Cathy
Diane
Betty
Cathy
Diane
Ann
Cathy
Diane
Ann
Betty
Diane
Ann
Betty
Cathy
(Ann, Betty)
(Ann, Cathy)
(Ann, Diane)
(Betty, Ann)
(Betty, Cathy)
(Betty, Diane)
(Cathy, Ann)
(Cathy, Betty)
(Cathy, Diane)
(Diane, Ann)
(Diane, Betty)
(Diane, Cathy)

How many different arrangements are there? How many different selections are there?

7. In counting arrangements in exercise 6 you should have counted (Cathy, Diane) as one arrangement and (Diane, Cathy) as another. Do these two arrangements involve the same girls? Should the two arrangements be counted as two different selections, or should they be thought of as different arrangements of one and the same selection?

8. In the problem we have just worked, how many times as many arrangements as selections are there?

9. The problem of determining the number of ways of seating four girls, two at a time, may also be solved with the aid of a box diagram. Copy the box diagram shown below and fill in the required information. Assume that the seat on the left is filled first.

Number of ways of filling each seat		**Number of arrangements**
Left	**Right**	
×	=	

Class Discussion 8b

1. Here is a more complicated problem. Suppose again that the same four girls—Ann, Betty, Cathy, and Diane—go to a tournament, but this time suppose that three seats are available. How many different seating arrangements are possible? Use a box diagram to find the answer to the question. Assume that the seat on the left is to be filled first, then the center seat, and finally the seat on the right.

Number of ways of filling each seat			**Number of arrangements**
Left	**Center**	**Right**	
×	×	=	

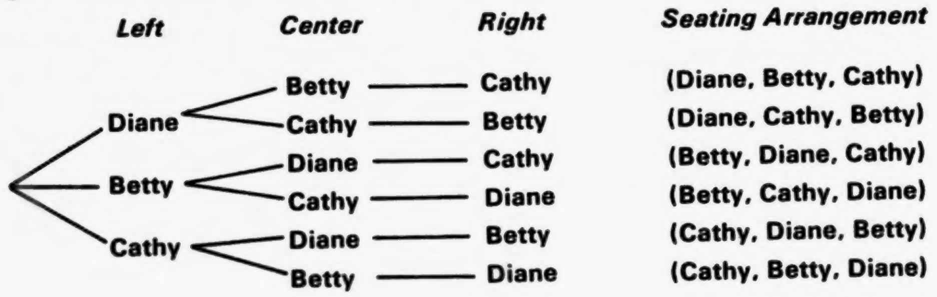
2. Your completed box diagram should appear as shown below. Note that the number of arrangements is 24. Would the result have been the same if the seat on the right had been filled first, the center seat next, and the seat on the left last?

Number of ways of filling each seat			**Number of arrangements**
Left	**Center**	**Right**	
4 ×	**3 ×**	**2 =**	**24**

3. We now ask, How many selections are there? Perhaps you are tempted to answer "12." This is not surprising in view of the problem we worked before. But, as we shall see, 12 is not correct.

4. You may have noticed that in displaying seating arrangements we have listed the names of the persons in the order in which they were seated from left to right and enclosed the list of names within parentheses ( ). To indicate when we are talking about selections, we shall make use of braces { }. That is, we shall use the notation {Ann, Betty, Cathy} to refer to a selection of three girls. {Betty, Cathy, Ann} represents the *same* selection. List all possible selections of three girls from the four who went to a tournament.

5. You should have listed four selections in exercise 4. Now let us take a different approach to the problem of determining the number of ways of selecting three girls out of four. Consider the selection {Diane, Betty, Cathy}. In how many ways can these three girls be arranged in three seats?


Left
Center
Right
Seating Arrangement
Diane
Betty
Cathy
Betty
Cathy
Diane
Cathy
Diane
Betty
Cathy
Betty
Cathy
Diane
Betty
Diane
(Diane, Betty, Cathy)
(Diane, Cathy, Betty)
(Betty, Diane, Cathy)
(Betty, Cathy, Diane)
(Cathy, Diane, Betty)
(Cathy, Betty, Diane)

6. How many arrangements are possible for the selection {Ann, Betty, Cathy}?

7. How many arrangements are possible for the selection {Ann, Cathy, Diane}?

8. How many arrangements are possible for any selection of three girls?

9. Knowing the number of ways of arranging four girls in three seats, by seating three at a time, how would you calculate the number of ways of selecting three girls from four girls?

Let us compare the present problem of seating four girls with three seats available to the previous problem, which involved the seating of four girls with two seats available. Recall that twelve arrangements were possible in the case of four girls and two seats. This was so because the girl to be seated in the left seat could be chosen in four ways, and after that the girl to be seated in the right seat could be chosen in three ways: $4 \times 3 = 12$. To find the number of possible selections of two girls to be seated, we divide 12 by 2. We do this because each selection of two girls has two possible arrangements. For example, the selection {Ann, Betty} has the arrangements (Ann, Betty) and (Betty, Ann).

Now let us look again at the problem which involves the seating of four girls in three seats when three girls are seated at one time. The total number of possible seating arrangements in this case is $4 \times 3 \times 2 = 24$. But each selection of three girls can be arranged in three seats in six ways. To determine the number of possible ways of selecting three girls from four girls, we divide the total number of possible arrangements, 24, by the number of possible arrangements of each selection; that is, $24 \div 6 = 4$. Hence, the number of ways of making a selection of three girls from four girls is 4.

Let us consider one more problem. Suppose that there are five people to be seated and three seats are available. The number of possible arrangements of five people taken three at a time is $5 \times 4 \times 3 = 60$. Do you see why? In any selection of three people the number of possible arrangements of the three people is $3 \times 2 \times 1 = 6$. In how many ways can three people be selected from five people? The answer is $60 \div 6 = 10$.

Summary—8

A general method of computing the number of possible ways of making a selection of p things from n things now suggests itself. We ask two key questions:

(1) What is the number of possible arrangements of n things taken p at a time?

(2) What is the number of possible arrangements of a selection of p things if all p things are used in each arrangement?

The answer to each question can be obtained by using either a tree diagram or a box diagram. To determine the number of ways of making a selection of p things from n things, divide the answer to (1) by the answer to (2).

The next set of exercises involves problems of selection. Use the general method of computation described above whenever applicable. Remember that if a selection consists of more than one thing, then the selection will have more than one arrangement.

Exercises—8

1. Three girls are eligible to go on a trip, but only one of the girls is to be given the opportunity. In how many ways can the selection be made?
2. Three girls are eligible for the girls' doubles team in tennis. In how many ways can the coach make a selection of one team?
3. A club has four members. In how many ways can a social committee of two be selected?
4. The library has four books that are of interest to Jim, but he is allowed to check out only three. How many selections of three books can he make from the four books in which he is interested?
5. Five girls are trying out for a girls' trio. In how many ways can the music teacher select a trio?

6. Roger is selecting trout flies for a fishing trip. He has only enough money to buy four flies out of the six that he likes. In how many different ways can he make a selection of four flies?

7. A contractor has job openings for three carpenters. If six carpenters apply, how many different selections of three men can the contractor make?

8. There are five candidates for the two end positions on a football team. In how many ways can the coach make a selection of two players from among the five candidates?

9. If a club has four members, in how many ways can a selection of a four-member committee be made?

10. In planning a trip a family needs to select three camping places. Seven places are available. In how many ways can a selection of three places be made?

11. Five boys are training to run in the mile relay. Four boys are to be selected. In how many ways can a selection of four boys be made?

12. There are eight boys on the basketball squad. In how many ways can the coach make a selection of the starting five?

13. Joan has narrowed her choice of sweaters down to six. If she can purchase only two, how many different selections of two sweaters can she make?

14. There are eight members in the student council. Three are to be appointed to the finance committee. How many different selections of a committee of three members are possible?

15. Seven boys are competing for the two guard positions on the basketball team. In how many ways can the coach make a selection of two guards?

9 A New Use for Pascal's Triangle

A 1965 penny has a profile of Lincoln on one side and a picture of the Lincoln Memorial on the other. The side with the profile of Lincoln is often referred to as "heads" and the other side as "tails." We shall represent heads by an unshaded circle ○ and tails by a shaded circle ●.

Now suppose that we put down one or more coins and that we are interested in the number of heads that face up. A single coin can have either no head or one head facing up. To enter this information in a table, we need a table with columns labeled "0 heads" and "1 head."

0 heads	1 head
1	1

Class Discussion 9

1. Consider the different arrangements that are possible with two coins.

○ ● ● ○
○ ○ ● ●

How many of the possible arrangements contain 0 heads? 1 head? 2 heads? Enter this information in a table like the one below.

0 heads	1 head	2 heads

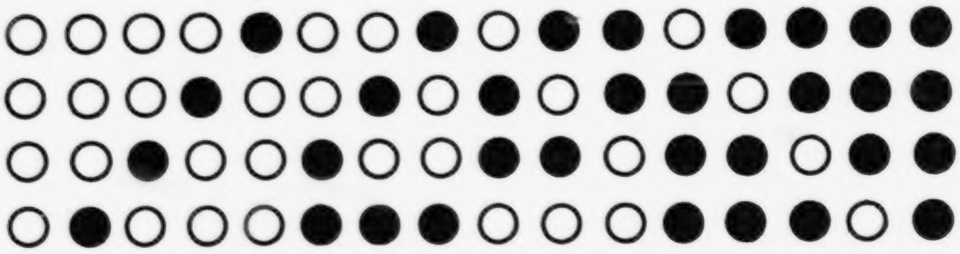
2. Now study the different arrangements that are possible with three coins.

● ○ ○ ○ ● ○ ● ●
● ○ ○ ● ○ ● ○ ●
● ○ ● ○ ○ ● ● ○

Since there are three heads in one of the arrangements, we need to extend the table shown in exercise 1 so that it will include a column labeled "3 heads." Copy and complete the table.

0 heads	1 head	2 heads	3 heads

3. Let's consider the different arrangements that are possible with four coins.



At least one of the arrangements will contain four heads, so a table like the one shown below is needed. Copy and complete the table.

0 heads	1 head	2 heads	3 heads	4 heads

4. Look again at the table in exercise 2. The sequence of numbers in the table is 1, 3, 3, 1. What is the sequence of numbers in the table in exercise 3?

5. Can you predict what the sequence of numbers in a table for five coins would look like?

6. Do the sequences of numbers in the tables displayed above suggest something you have seen before?

7. Now let us return to the problem of selecting a committee from a group of persons.

If a club has three members, in how many ways can you select a committee with only one member? In how many ways can you select a committee of two members? In how many ways can you select a committee of three members? (To obtain your answers, use the method of counting selections that you learned in Section 8.) Enter your answers in a chart like the one below.

Committee of		
one	two	three

You should have obtained the numbers 3, 3, 1 for your answers. The numbers in the first two spaces in your chart should agree with the answers to exercises 1 and 2 on page 33. Explain why.

Exercises—9

1. Consider a club of four members.
  - **a.** In how many ways can a committee of one member be selected?
  - **b.** In how many ways can a committee of two members be selected?
  - **c.** In how many ways can a committee of three members be selected?
  - **d.** In how many ways can a committee of four members be selected?

2. For a club of five members, find the number of ways in which a committee can be selected if the committee is to have—
  - **a.** One member.
  - **b.** Two members.
  - **c.** Three members.
  - **d.** Four members.
  - **e.** Five members.

3. Do the sequences of answers in exercises 1 and 2 suggest a familiar pattern with, perhaps, something left out? With this in mind, find the number of ways in which committees of various sizes can be formed in a club that has six members.

4. Let us return now to the coin problems. Complete the array of numbers started below by writing down the entries from the coin tables in a way that follows the pattern that is started.

Table for one coin	1 1
Table for two coins	1 2 1
Table for three coins	
Table for four coins	
Table for five coins	

5. It is not hard to see that the array in exercise 4 is related to Pascal's triangle. Note, however, that the 1 at the top is missing. Extend the number array that you started in exercise 4 far enough to include the number 252.

6. To answer the questions below, you will need to make use of the information you were asked to supply in exercise 4 and the array you were asked to produce in exercise 5.

a. Which row refers to the placement of five coins?
b. Which row refers to the placement of six coins?
c. Which row refers to the placement of seven coins?
d. Which row refers to the placement of ten coins?

7. Is the selection of committees from the members of a club also related to Pascal's triangle? Explain your answer.

8. There is a slight difference between the number arrays for coins and those for committees. For example, the third row in the array for committees is 3, 3, 1. That is, the sequence in the third row of the array for the committees starts with the second number of the corresponding sequence for coins. This same discrepancy exists for all pairs of corresponding rows. If we wanted both arrays to be the same, we would have to change the committee selection problems to include the possibility that a committee may have no members. By way of illustration let us consider this possibility in selecting committees from a club that has five members. How many committees can be formed that will contain no members? The question sounds strange, but let us agree that a sensible answer is "One." In accordance with this agreement, a table showing the number of committees of various sizes that can be selected from a club having five members would appear as shown below.

Committee of					
no members	one	two	three	four	five
1	5	10	10	5	1

In how many ways can committees of various sizes be selected from a club that has six members? Be sure to include the committee that has no members. Record your answers in a table similar to the one above.

9. Make a table that indicates the number of ways in which committees of various sizes can be selected from a club of seven members; eight members; nine members; ten members. In each case be sure to include the committee that has no members.

10. If a committee having no members is counted as one selection in each committee-selection problem, then the complete set of tables for committees will look like the set of tables for coins. Both sets of tables are closely related to Pascal's triangle. With this in mind, find an easy way to answer the following questions:

a. In how many ways can a committee of three members be selected from a club that has eight members?

b. In how many ways can a committee of four members be selected from a club that has ten members?

c. In how many ways can a committee of six members be selected from a club that has nine members?

d. In how many ways can a committee of two members be selected from a club that has six members?

e. In how many ways can a committee of three members be selected from a club that has nine members?

10 Probability

At the beginning of a football game a coin is tossed to determine which team will have the choice to kick or receive. It is assumed that the coin used is "honest" and that the toss is "fair." This means that the chance of the coin's landing with heads showing is the same as the chance of its landing with tails showing. Now suppose we ask the question, What is the probability that when the coin is tossed it will show heads? Since there are two equally likely ways in which the coin can land, and in one case the coin will show heads, we say that the probability of heads is $\frac{1}{2}$.

Now let us consider another situation. Suppose that a box contains five slips of paper of the same size and shape, that one of these slips is marked with an X, and that the remaining four slips are blank. You are to draw a slip without looking. Since there are five slips, there are five possible outcomes. Assume that all possible outcomes are equally likely. What is the probability that you will draw the slip with an X? Reasoning as before, we can state that the probability of drawing the slip marked with an X is $\frac{1}{5}$.

Class Discussion **10a**

1. Assume that two of the five slips described above are marked and that the remaining three are blank. What is the probability that you will draw a marked slip if, as before, you are to draw one slip?

2. Suppose that there are ten slips of paper, that three of the slips are marked, and that the remaining seven are blank. You are to draw one slip without looking. Assume as before that any slip has the same chance of being drawn as any other. What is the probability of drawing a marked slip?

3. What is the number of possible outcomes in exercise 2?

4. If we call drawing a marked slip a successful outcome, how many of the possible outcomes in exercise 2 may be called successful?

5. Your answer to exercise 2 should be $\frac{3}{10}$. Note that the denominator of this fraction is the number of possible outcomes, and that the numerator is the number of possible successful outcomes. These facts suggest the following definition of probability. *If there are a certain number of possible outcomes in an ex-*

periment, if all these outcomes are equally likely, and if some of these outcomes are considered successful; then the probability of a successful outcome can be expressed by a fraction whose denominator is the number of possible outcomes and whose numerator is the number of possible successful outcomes. If we have 12 slips of paper and 5 of these are marked, what is the probability of drawing a marked slip?

6. In the case of tossing a single "honest" coin we noted that there were two possible outcomes—heads and tails. If we consider tails to be a successful outcome, what is the probability of obtaining a successful outcome in one toss of an "honest" coin?

7. Suppose that a box contains 15 marbles. Six are blue and the rest are green. All marbles are the same size. The box is shaken so that the marbles are well mixed. You are to draw a marble out of the box without looking. What is the probability that you will draw a blue marble in one draw?

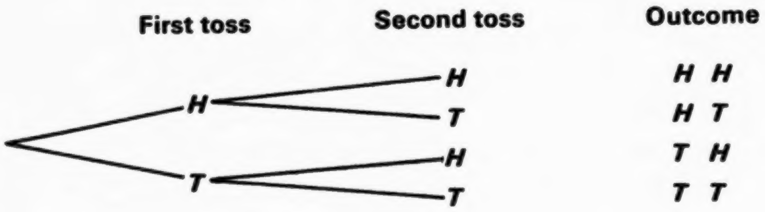
8. What is the probability that you will draw a green marble in one draw?

In finding the probability in each situation considered thus far, we assumed that all possible outcomes were equally likely. It is important to remember that this assumption is part of the definition of probability as stated in exercise 5. This does not mean, of course, that outcomes actually are equally likely. If, in exercise 7, the blue marbles are put in the box last and the marbles are not well mixed, then all possible outcomes will probably not be equally likely.

Let us now return to some of the ideas introduced in previous sections and consider them again in connection with the notion of probability.

Class Discussion **10b**

1. If a coin is tossed two times in succession, what are the possible outcomes? We can use methods we have mastered to help us find the answer.


First toss
Second toss
Outcome
H
T
H
T
H
T
H H
H T
T H
T T

2. In how many of the possible outcomes are the results of both tosses tails? What is the probability that this will happen?

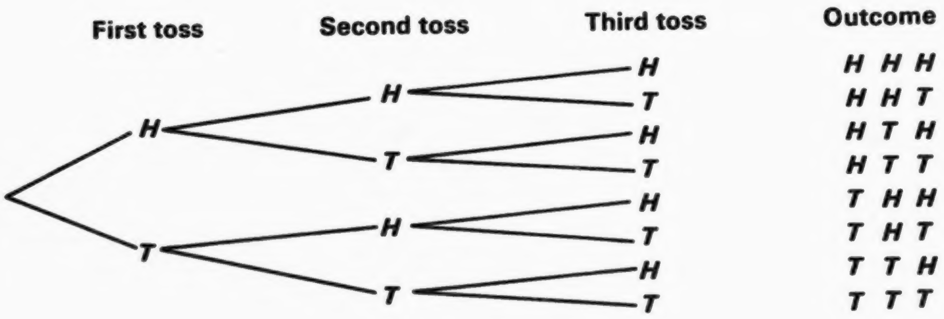
3. What is the probability that one head and one tail will turn up in two tosses?

4. One of the four possible outcomes is that the coin will turn up heads in both tosses. What is the probability that this will happen?

A table listing the probabilities asked for in exercises 2–4 is given below.

Number of heads	Probability
0	$\frac{1}{4}$
1	$\frac{1}{2}$
2	$\frac{1}{4}$

5. If a coin is tossed three times in succession, the number of possible outcomes is increased. How many possible outcomes are there?


First toss
Second toss
Third toss
Outcome
H
T
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H H H
H H T
H T H
H T T
T H H
T H T
T T H
T T T

6. In how many of the possible outcomes of three tosses of one coin are there only heads?
7. What is the probability of obtaining three heads in three tosses of one coin?
8. What is the probability of obtaining two heads and one tail in three tosses of one coin?
9. What is the probability of obtaining one head and two tails in three tosses of one coin?
10. Make a table listing the probabilities of obtaining 0, 1, 2, or 3 heads when one coin is tossed three times in succession. (See exercise 4.)

Exercises—10

1. Construct a tree diagram to illustrate the possible outcomes of an experiment that involves tossing one coin four times in a row.
2. How many possible outcomes are there?
3. In how many of the possible outcomes are there three heads?
4. What is the probability that exactly three heads will appear in four tosses of one coin?
5. Use the table that you made in exercise 3 of Class Discussion 9 to determine the probability of getting exactly two heads in four tosses of a coin.
6. In this exercise we shall make use of Pascal's triangle to determine probability. If a coin is tossed six times, we can use the row in Pascal's triangle containing the numbers 1, 6, 15, 20, 15, 6, 1 to find the number of possible outcomes that have a given number of heads. For example, there is one possible outcome having no heads, six having one head, fifteen having two heads, and so on.
   a. What is the total number of possible outcomes?

b. If a coin is tossed six times, what is the probability that heads will turn up exactly four times?
c. If a coin is tossed six times, what is the probability that heads will turn up exactly five times?

7. If seven coins are tossed, what is the probability that five coins will turn up heads? (*Hint:* Use Pascal's triangle.)

8. A box contains 13 slips of paper of which five are marked. The slips have been well mixed. A drawing is to be made without looking. What is the probability that an unmarked slip will be drawn?

9. A bowl contains 20 marbles. Four of the marbles are green, six are blue, and the remainder are black. The marbles are well mixed. You are to draw one marble without looking. What is the probability that you will draw a green marble in a single draw? A blue one? A black one?

10. A box contains 26 disks. Each disk is marked with a different letter of the alphabet. The disks are well mixed. What is the probability of drawing a disk marked with a vowel in a single draw? What is the probability of drawing a disk marked with a letter that comes after Q in a single draw?

11 Exploring Unknown Probabilities

The last section dealt with questions of probability in experiments that involved the tossing of coins and the drawing of objects from a box. In each case we determined the number of *possible outcomes* of the experiment, and also the number of *possible successful outcomes*. On the assumption that all possible outcomes were *equally likely*, we expressed the probability of a successful outcome by a fraction with the number of possible outcomes as the denominator and the number of possible successful outcomes as the numerator. This was in accordance with the definition of probability stated in exercise 5 of Class Discussion 10a. A probability determined in this way is referred to as a *theoretical probability*. It is important to note that a theoretical probability is based on a definition, and not on the actual results of an experiment.

In many situations we cannot make the assumption that all outcomes are equally likely. Consider, for example, an experiment that involves the tossing of a thumbtack. Would anyone be willing to say that the chance of the tack's landing with the point up is the same as the chance of its landing with the point down? Common sense tells us that these outcomes are not equally likely. To answer questions about probability in cases like this, we often have to study the *actual results* of experiments. What we do is this. We conduct a reasonably large number of repeated trials. We keep track of the number of trials, and we count the number of outcomes that we have agreed to call successful. Then we use a fraction whose denominator is the number of trials and whose numerator is the number of successful outcomes to *estimate* the probability of a successful outcome. (The word "estimate" is all-important here, because each set of repeated trials may give us a different fraction.) A probability arrived at in the manner described is referred to as an *experimental probability*.

Consider again the experiment that involves the tossing of a thumbtack. Suppose that the tack is tossed 25 times and that it lands point up five times. On the basis of this set of trials, we would then estimate the probability of the tack's landing point up as $\frac{5}{25}$ or $\frac{1}{5}$. What would you give as an estimate of the probability that the tack will land point up the next time it is tossed if in 50 trials it lands point up 11 times?

As another example, suppose that we were asked what the probability is that a certain baseball player will get a hit on his next time at bat. To estimate this probability, we look at the player's record. If he has been at bat 200 times and has 50 hits, what would you give as an estimate of the probability that he will get a hit on his next time at bat?

Class Discussion 11

Review the discussion concerning the tossing of a toothpaste-tube cap in Section 2. Recall that we used the letter *S* to represent the case in which the cap comes to rest on its side, the letter *B* to represent the case in which it comes to rest on its bottom, and the letter *T* to represent the case in which it comes to rest on its top.

1. Do you think the probability of *T* is greater than that of *B*?
2. Do you think the probability of *B* is greater than that of *S*?
3. Arrange *S*, *B*, and *T* in order, beginning with the one that you think has the greatest probability.
4. Guess what the probabilities of *S*, *B*, and *T* might be.
5. How good do you think the guesses are that you made in exercise 4? You can find out by doing an experiment.

   Supply yourself with a toothpaste-tube cap and prepare a chart like the one below. According to the chart, you are to toss the cap ten times in the first set of trials. Record the outcome for each toss, using the letter *S*, *B*, or *T* as the case may be. Count the *S*'s, *B*'s, and *T*'s that you recorded for the ten tosses and indicate the number of each in the last three columns of the chart. Repeat this procedure for each set of trials. Then find the total for each of the last three columns.

Toss number	1	2	3	4	5	6	7	8	9	10	No. of times outcome was		
											S	*B*	*T*
First set of trials													
Second set of trials													
Third set of trials													
Fourth set of trials													
Fifth set of trials													
Sixth set of trials													
Seventh set of trials													
Eighth set of trials													
Ninth set of trials													
Tenth set of trials													
Totals													

a. How many times did you toss the cap to complete all ten sets of trials?

b. How many times did the cap come to rest on its side in the ten sets of trials?

c. What would you give as an estimate of the probability of *S*? Of *B*? Of *T*?

d. The estimates you gave in exercise 5c are experimental probabilities. Why are they called "experimental"?

e. How do your estimates of the probabilities of *S*, *B*, and *T* compare with the guesses you made in exercise 4?

f. How many times would you predict that the cap would come to rest on its top if you were to toss the cap 20 times more?

g. Toss the cap 20 times and compare the actual result with the prediction that you made in exercise 5f.

General Summary

In this unit we have considered several of the big ideas connected with arrangements, selections, and probability.

1. To determine the number of possible arrangements of a group of n things taken p at a time, we made use of the fundamental principle of arrangements and displayed the answer by means of a tree diagram and a box diagram.
2. To determine how many selections of p things can be made from a group of n things, we developed a computational procedure that may be described by the formula

   $$S = A \div N.$$

   In this formula, S represents the number of possible selections of p things from n things. A represents the number of possible arrangements of n things taken p at a time. Finally, N represents the number of possible arrangements of a selection of p things when all p things are used in each arrangement.
3. In problems that involve arrangements, the order or placement of things counts. In a selection, order does *not* count.

4. The triangular array of numbers known as Pascal's triangle is valuable as a reference in answering the question of how many ways a selection of a given number of things can be made from a group.

5. Probability can be thought of as a measure of the chance that a certain outcome will occur. If all possible outcomes of an experiment are equally likely, then the probability of a successful outcome can be expressed by a fraction whose denominator is the number of possible outcomes and whose numerator is the number of possible successful outcomes.

6. In Section 10 we dealt with questions of probability in experiments that involved the tossing of coins and the drawing of slips of paper, marbles, and disks from a box. In each case we determined the probability of a successful outcome by counting the number of possible outcomes and the number of possible successful outcomes. Then, on the assumption that all possible outcomes were equally likely, we expressed the probability of a successful outcome by a fraction. (See statement 5 above.) In none of the experiments with coins, slips of paper, marbles, or disks did we actually carry out an experiment. In each case we simply stated a probability on the basis of a definition. When a probability is determined in this way, it is referred to as a theoretical probability.

7. In Section 11 we noted that outcomes in certain experiments are not equally likely, and furthermore that it is not sensible to assume that they are. It was pointed out that in cases like this it is often necessary to conduct a set of trials and to estimate the probability of a particular outcome on the basis of actual results. In the experiments described it was stated that the probability of a particular outcome could be estimated by a fraction whose denominator was the number of trials and whose numerator was the actual number of outcomes of the kind being observed. When a probability is arrived at in this manner, it is referred to as an experimental probability.

8. The idea of probability is used in a variety of situations. What

is the chance that you will be in an automobile accident during the coming year? What is the probability that you will reach the age of seventy-five? Answers to questions like the foregoing are of vital concern to accident- and life-insurance companies.

Review Exercises

1. Four coins are tossed. Make a display showing all possible results. Let *T* represent tails, and let *H* represent heads.
2. A certain school uses only two-digit numerals on its football uniforms. How many uniforms can be labeled differently if only the digits 1, 3, 5, 7, and 9 are used?
3. A recreation center has eight bowling alleys. In how many ways can eight teams be assigned to these alleys?
4. Build a number array in the form of Pascal's triangle, entering numbers through the row in which the number 35 appears for the first time.
   - **a.** What is the fourth number from the left in the seventh row?
   - **b.** This number answers the question: How many different committees of________members each can be formed from a group of_____people?
   - **c.** What is the sum of the numbers in the seventh row?
   - **d.** If an "honest" coin is tossed six times, what is the probability that heads will turn up exactly three times?
5. In how many ways can six students line up for tickets to a school play?
6. In how many ways can a girl choose two different dresses from a rack of twelve dresses?
7. How many different committees of three members each can be chosen from a group of six people?
8. At a high school track meet five boys are to be lined up to run the 100-yard dash. In how many ways can the five boys be lined up on the starting line?

9. Arrange an elimination-tournament schedule for 16 teams.
  a. How many rounds of play are needed?
  b. How many games should be scheduled altogether?
  c. If 10 more teams enter the tournament at the last minute, how many rounds will have to be played?
  d. With 26 teams entered, how many "byes" must there be if all "byes" are taken in the first round?

10. Six boys tried out for the two male roles in the senior-class play. How many different selections of two male actors can be made?

11. In how many different ways can a family of five be seated on a bench?

12. A box contains 22 slips of paper. Five of the slips are marked X, seven are marked Y, and the rest are blank. The slips are well mixed. You are to draw a slip without looking. What is the probability that you will draw a blank slip in one draw?

13. An "honest" coin is tossed four times. What is the probability that heads will turn up exactly two times?

14. A dime store has found that, out of an order of 1,000 ball-point pens, 15 are usually defective. What is the probability that a customer who buys a ball-point pen will find it defective?

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Answers for Units 1-5

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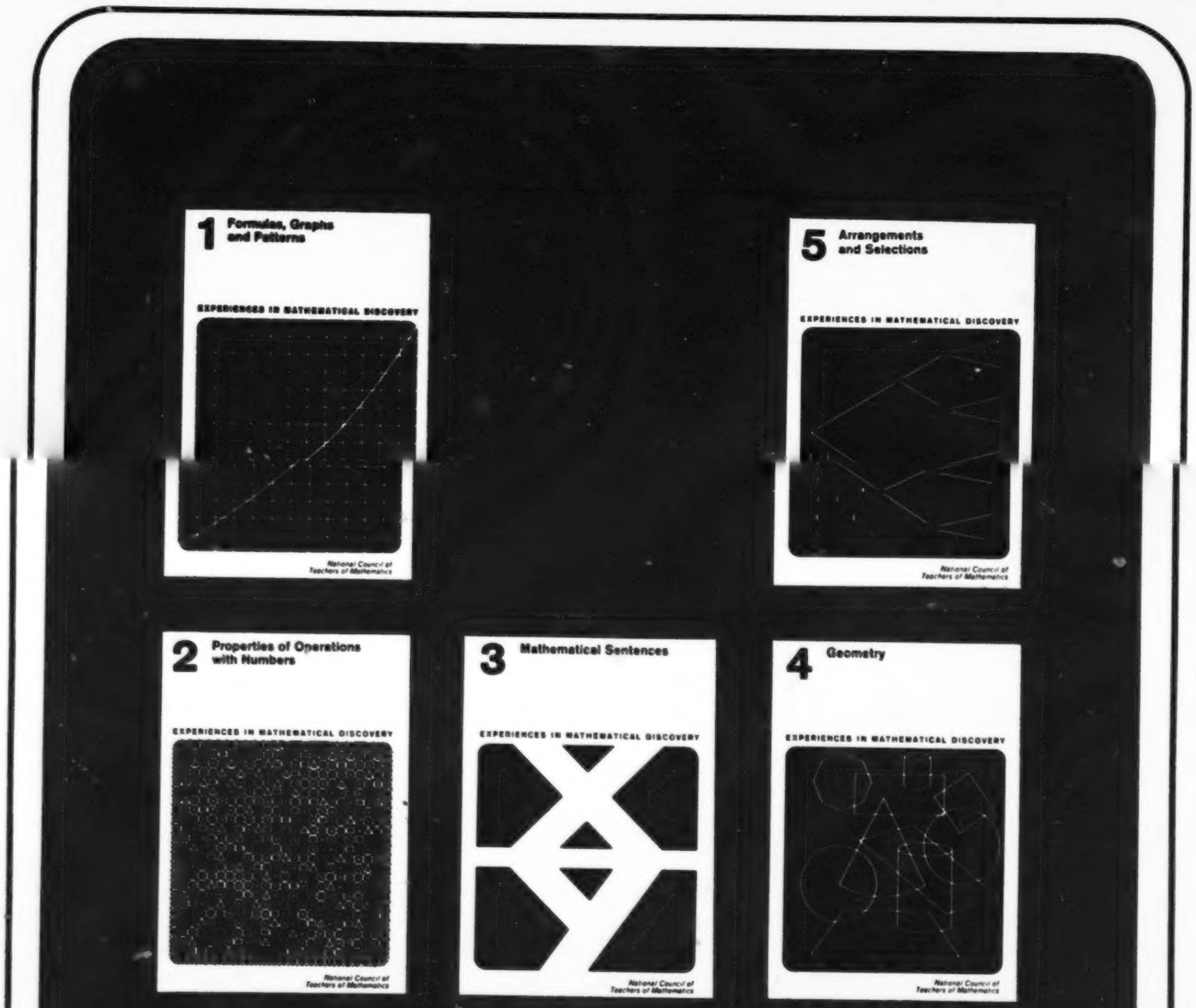
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ANSWERS FOR UNITS 1-5

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ANSWERS FOR

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Unit 5. Arrangements and Selections 109

UNIT 1 ANSWERS

Pages 2-4 Class Discussion-1

1. Yes
2. Yes * Second shot, (4,2)
3a. No
b. Yes
c. Yes * First shot, (4,5)
4a. Yes * David should know that one part of the ship is at (2,1), (4,2), or (1,1), and he should have tried adjacent points.
b. Either by listing ordered pairs or by marking them on a grid.
5a. Yes * They are on the same horizontal line as (4,5) and within two points of it.
b. No
6. (3,2) and (5,2)
7. 4 and 5; 4 and 6; 4 and 7
8. No
9. Horizontal axis: (5,0), (0,0), (2,0) * Vertical axis: (0,6), (0,1), (0,0) * Neither axis: (3,3), (4,1)
10. 64 ordered pairs
11. Horizontal line * Yes. The second coordinate is the same in each ordered pair. This means that the points will be the same distance above the horizontal axis.
12. His chances are poor. There are too many possible choices.
13. An infinite number

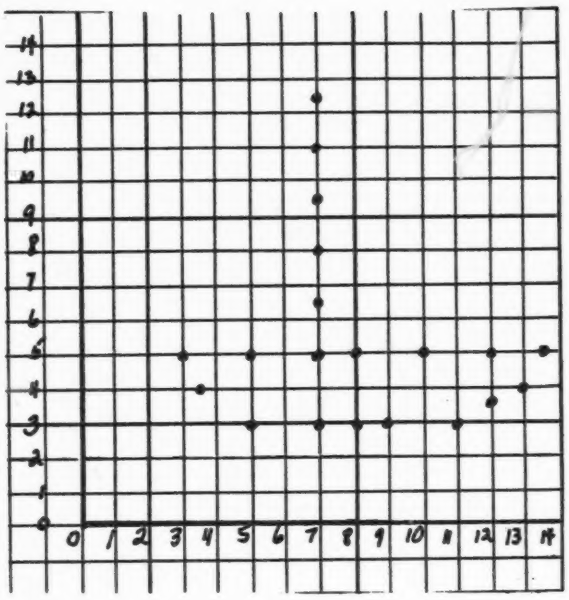
Pages 5-8 Exercises-1

1. Point A, (2,8)
Point B, (4,9)
Point C, (6,8)
Point D, (9,7)
Point E, (3,6)
Point F, (3,3)
Point G, (5,4)
Point H, (7,5)
Point K, (5,1)
Point L, (8,3)
Point M, (1,1)
Point N, (10,10)

Exercises-1 (continued)

2. The pattern suggests a sailboat. * Each number is 3 greater than the preceding number. *

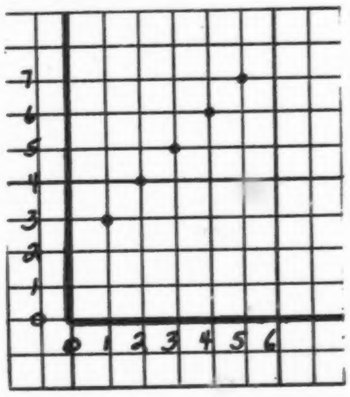
Graph



3a. (1,1), (3,3), (7,7), (9,9)
b. A line
c. Any examples of the form (a,a), such as (5,5)

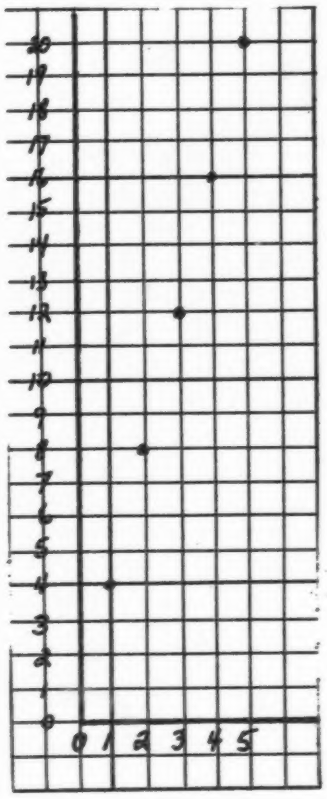
4. Five possible examples in the form (a-2,a) are (1,3), (2,4), (3,5), (4,6), (5,7) *

Graph



5. Five examples in the form (a,4a), such as (1,4), (2,8), (3,12), (4,16), (5,20) * Graph

(5)



6a. Five examples of the form (a,a-1), such as (1,0), (2,1), (3,2), (4,3), (5,4)

Exercises-1 (continued)

6b. Five examples such as (11,10), (12,11), (13,12), (14,13), (15,14)

7a. Use the horizontal and vertical axes as before. Place the third axes pointing out of the paper, directly up.

b. 3 units to the right of origin, 4 units up on the paper, and 7 units out from the paper

c. 5 units to the right of origin, 2 units up on the paper, 11 units out from the paper

Pages 8-10 Class Discussion-2

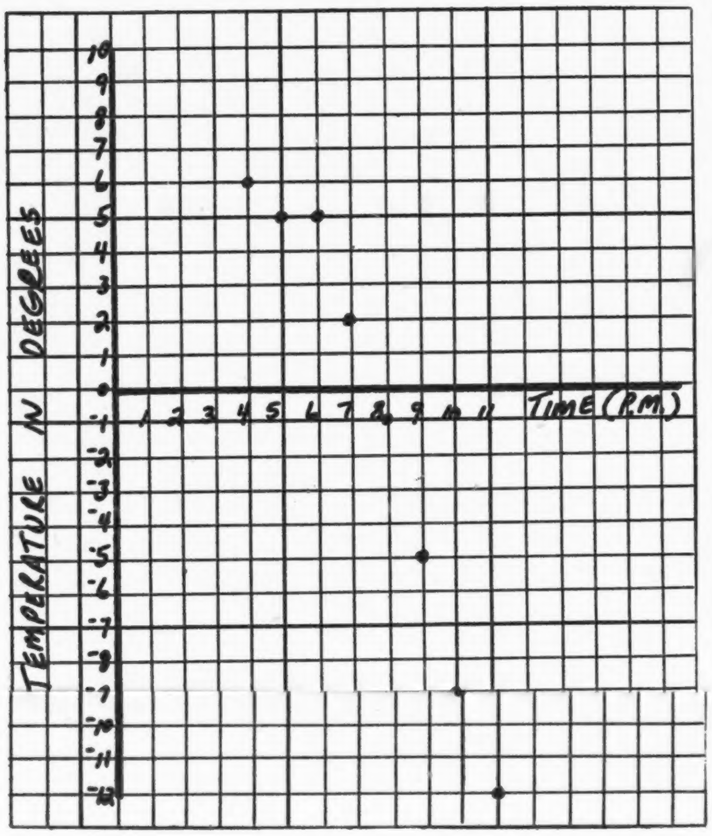
1. 7:00 P.M.
2. 4° below zero
3. -8°
4.

Time	4:00 P.M.	5:00 P.M.	6:00 P.M.	7:00 P.M.	8:00 P.M.	9:00 P.M.	10:00 P.M.	11:00 P.M.
Temperature	6°	4°	2°	0°	-2°	-4°	-6°	-8°

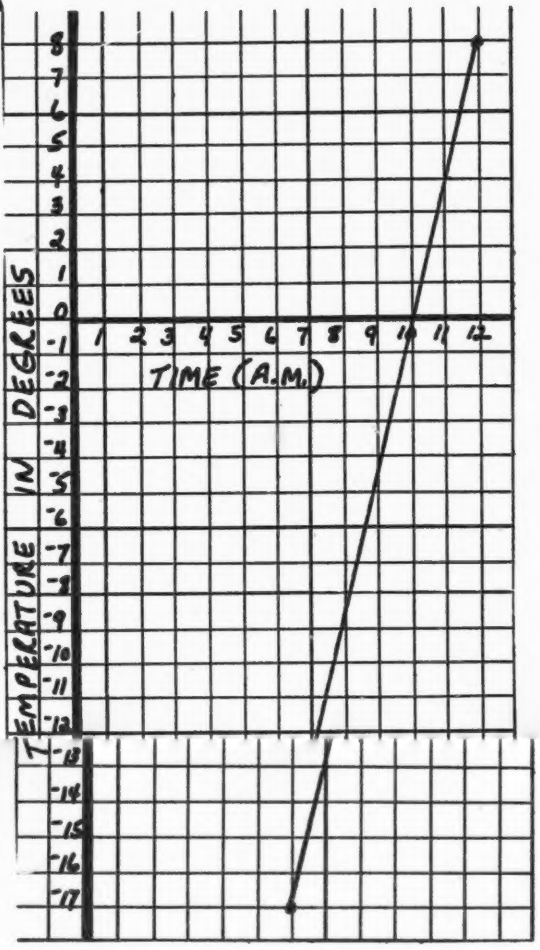
5. [illegible]
6. 8 units to the right of the vertical axis and 2 units below the horizontal axis
7. (4,6), (5,4), (6,2), (7,0), (8,-2), (9,-4), (10,-6), (11,-8) * Yes
8. 3°; 3-1/2°; 2-1/2°
9. Draw a line through the points on the graph. Then read the temperatures which correspond to times of 5:30, 5:15, 5:45 * Yes, they agree.
10. Yes * A line
11. No, the announcer's prediction was only for temperatures after 4:00 P.M. They may not have been falling before 4:00.

Pages 11-14 Exercises-2

1. Graph



(2A)



2a. Graph

b.

Time	Temperature (exact)
7:00 A.M.	-12 5/6°
8:00 A.M.	-8 2/3°
9:00 A.M.	-4 1/2°
10:00 A.M.	-1/3°
11:00 A.M.	3 5/6°

(Students' answers will be approximate.)

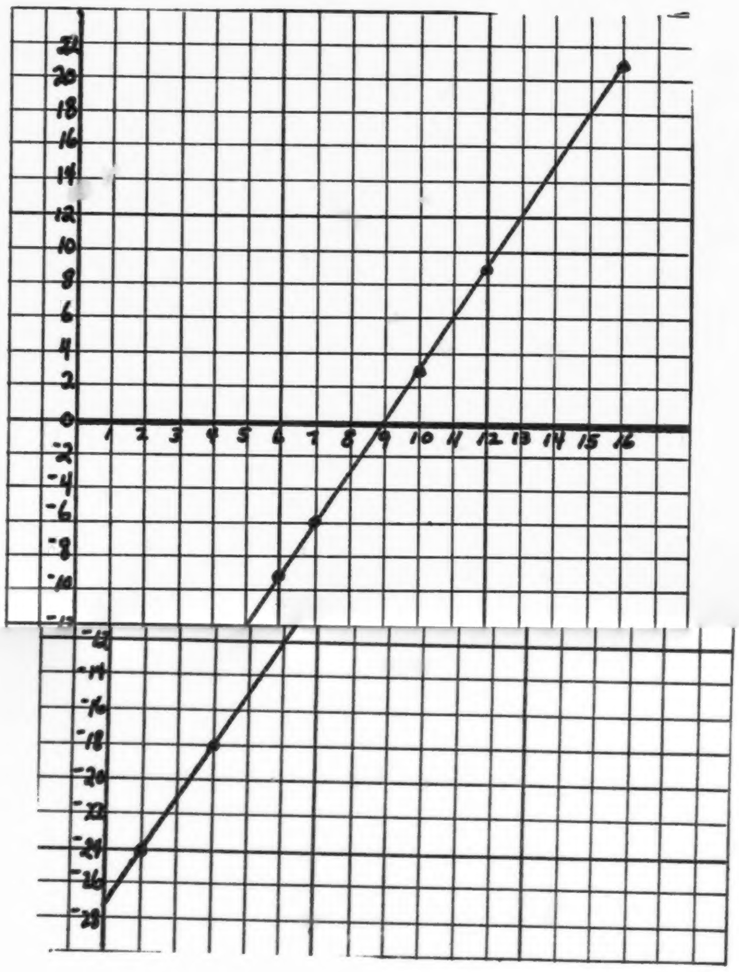
c. 7:02 A.M. * 9:53 A.M. * 10:05 A.M. * 11:31 A.M. (Students' answers will be approximate.)

3a. (1,1), (2,0), (3,-1), (4,-2), (5,-3)

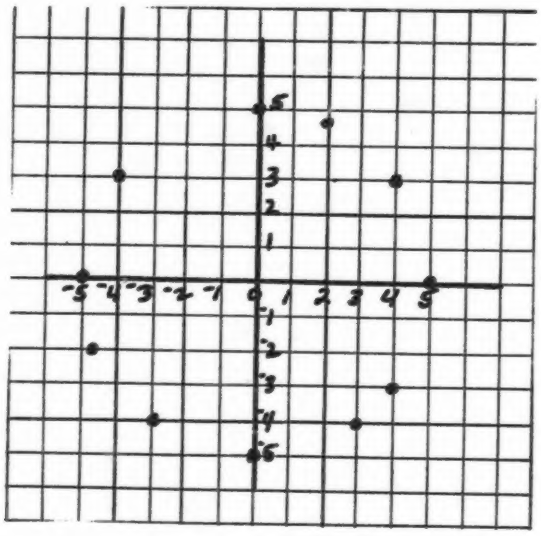
b. (1,1), (2,0), (3,-1), (4,-2), (5,-3)

Exercises-2 (continued)

3c. The same ordered pairs are plotted in both graphs.
d. The relative sizes of the units in the horizontal and vertical scales are different.
4. Graph
5. (9,0) * (0,-27)
6. Ship A, (5,6), (6,6), (7,6) *
Ship B, (-1,2), (0,2), (1,2) *
Ship C, (-5,1), (-5,2), (-5,3) *
Ship D, (-1,-4), (0,-4), (1,-4) *
Ship E, (-7,-5), (-7,-6), (-7,-7)

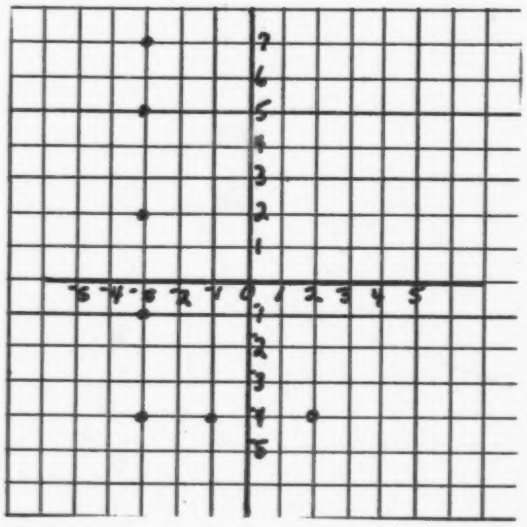


7a. Graph
b. Circle
8a. Left
b. Below



Exercises-2 (continued)

9. Graph * The letter is L. * Any five of these pairs: (-3,6), (-3,4), (-3,3), (-3,1), (-3,0), (-3,-2), (-3,-3), (-2,-4), (0,-4), (1,-4)

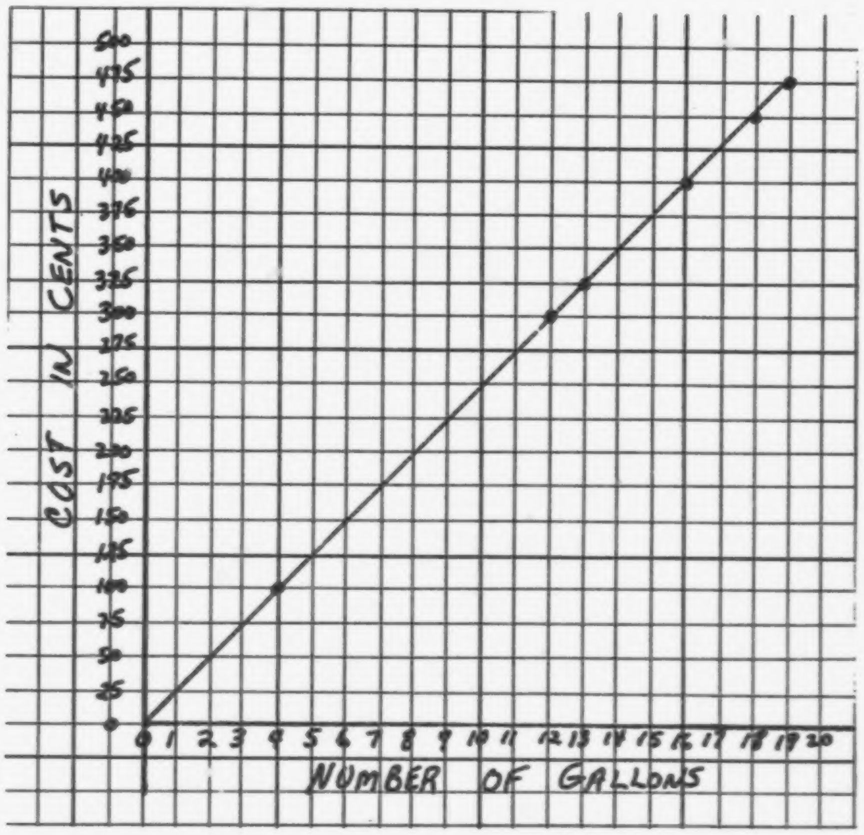


Pages 15-16 Class Discussion-3

1. 1 gallon
2. No. For example, between 0 and 100 each segment represents 25 cents. Between 100 and 300 each segment represents 50 cents.
3. 125 cents
4. No
5. The cost of 12 gallons is represented by twice as many units as the cost of 4 gallons.
6. The number of cents represented by one unit of cost at the bottom of the scale is generally greater than at the top. Thus the difference in cost between 0 and 13 gallons is represented by the same number of vertical units as the smaller difference in cost between 13 and 19 gallons.
7. No * A given amount of money is represented by a greater number of units at the top of the vertical scale than at the bottom of the scale.
8. The cost scale is not uniform.
9. Let each unit on the cost scale represent 25 cents (or some other uniform number of cents).

Pages 16-18 Exercises-3

1. Graph



2.

Number of Gallons	5	15	7	11	17	9	3	1
Cost in Cents	125	375	175	275	425	225	75	25

3a. 263 cents
b. 330 cents
c. 488 cents
(These are exact. Students' answers will be approximate.)
4a. 3.6 gal.
b. 9.4 gal.
c. 20.4 gal.
(Students' answers will be approximate.)
5. Answers vary. Some possibilities are (8,200) and (10,250)
6. Not possible
7. C = 25 x G.
8. On the graph, each time the line passes through one unit (1 gallon) of gasoline, it passes through 1 unit (25 cents) of cost.
9. C = 31 x G.

Pages 19-21 Class Discussion-4

1. Answers vary. For example, let G represent a number of yards of ribbon priced at 30 cents per yard and C represent the cost in cents of the ribbon. In this example, G need not be a whole number.

2a. 0
b. 30
c. 60
d. 120
e. 210
f. 270

3.

G	0	1	2	4	7	9
C	0	30	60	120	210	270

4. Yes. Any pairs of the form (a,30a)
5. Answers vary. For example, (1,10) * No
6a. Yes
b. Yes * Yes * No
c. Yes * Yes * Yes * If G had been replaced by 15, C would have been replaced by 450.
d. Yes
e. No * No * No * No

7.

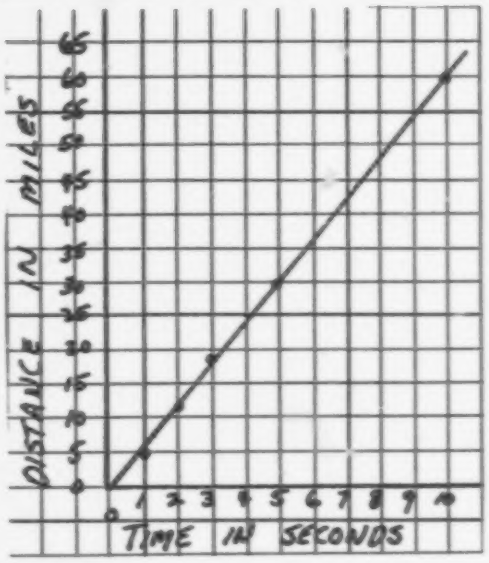
G	3	6	8	11	12	20
C	90	180	240	330	360	600

Pages 21-23 Exercises-4

1a. 6 miles * 12 miles * 60 miles
b.

t	0	1	2	3	5	7	10
D	0	6	12	18	30	42	60

c. Graph
d. 24 miles * 36 miles * 48 miles * 66 miles
e. 1-1/2 seconds * 3-1/2 seconds * 9 seconds
f. 15 seconds * 25 seconds * 50 seconds * 37-1/2 seconds



Exercises-4 (continued)

2a.

N	0	1	3	4	10	15	21	30	$33\frac{1}{3}$
C	0	3	9	12	30	45	63	90	100

b.

B	0	5	15	27	33	41	90	161	191
A	10	15	25	37	43	51	100	171	201

c.

T	15	25	35	50	80	100	115	215
R	0	10	20	35	65	85	100	200

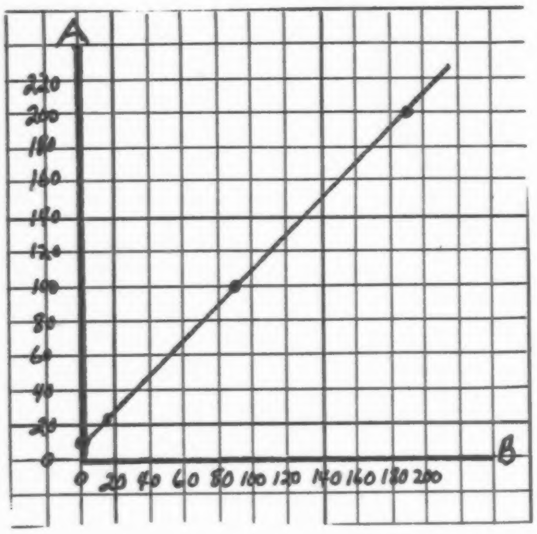
d.

A	10	20	40	70	80	100
X	1	2	4	7	8	10

e.

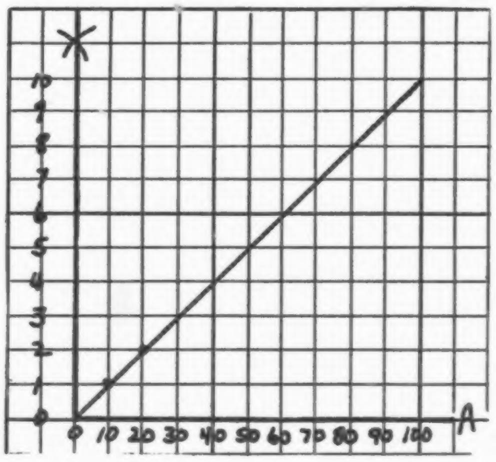
A	10	20	40	70	80	100
X	1	2	4	7	8	10

3a. Graph



Exercises-4 (continued)

3b. Graph



c. Same graph as in 3b

4. 21-45/67 lb. (about 22 lb.)

Pages 25-27 Class Discussion-5

1. 2 miles

2.

Time in Seconds (s)	1	3	5	8	10
Distance in Miles (D)	$\frac{1}{5}$	$\frac{3}{5}$	1	$1\frac{3}{5}$	2

3a. $\frac{1}{5}$

b. $\frac{3}{5}$

c. 1

d. $1\frac{3}{5}$

e. 2

f. Same as table in problem 2

4a. Yes

b. $y + 10 = x$.

5. C

6. Answers may vary. One possibility is: on the vertical axis 1 unit represents 1; on the horizontal axis 1 unit represents 5.

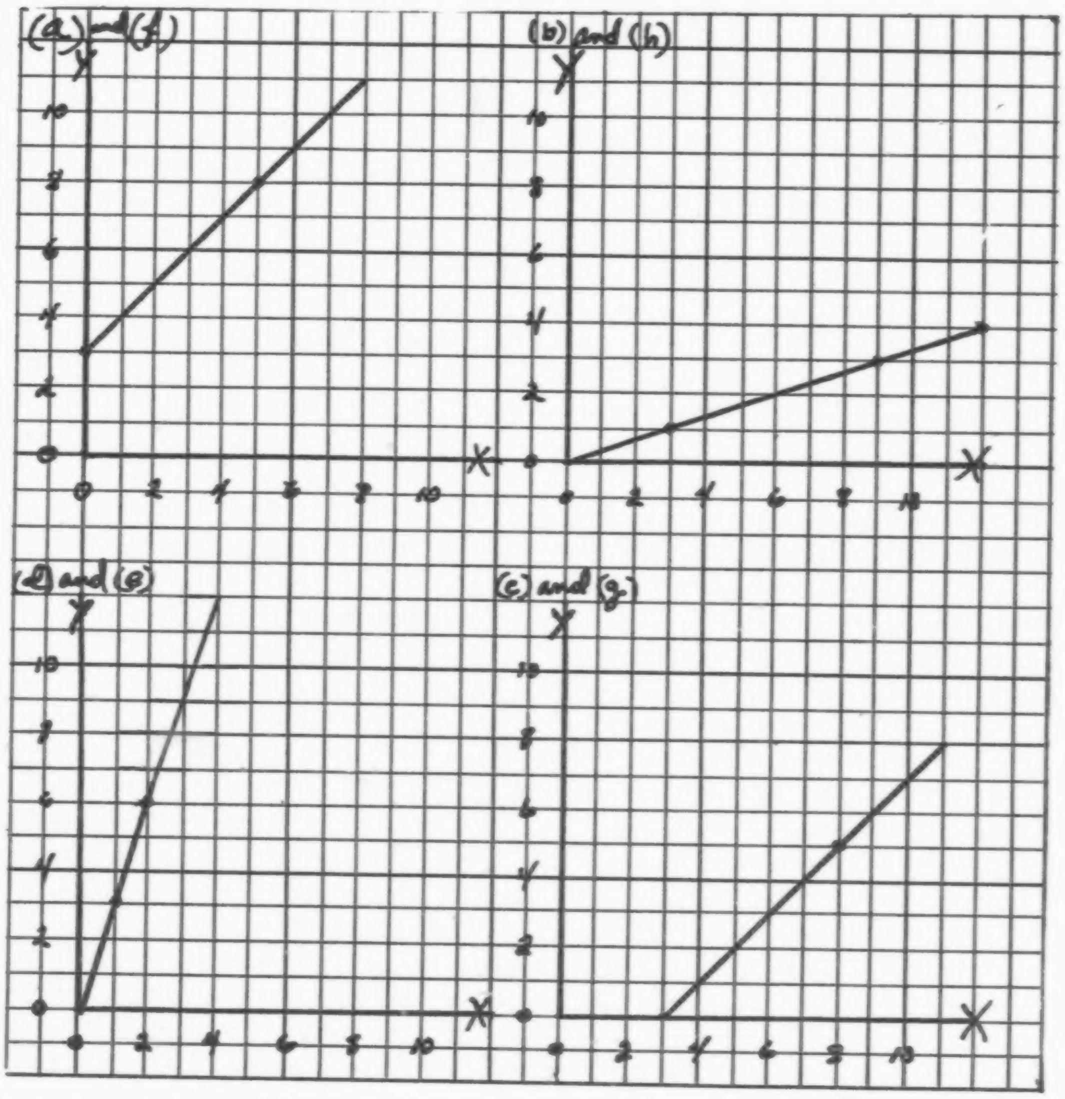
Class Discussion-5 (continued)

7a. If D becomes greater as s becomes greater the graph rises to the right. If D becomes smaller as s becomes greater the graph falls to the right.

b. See answer to 7a.

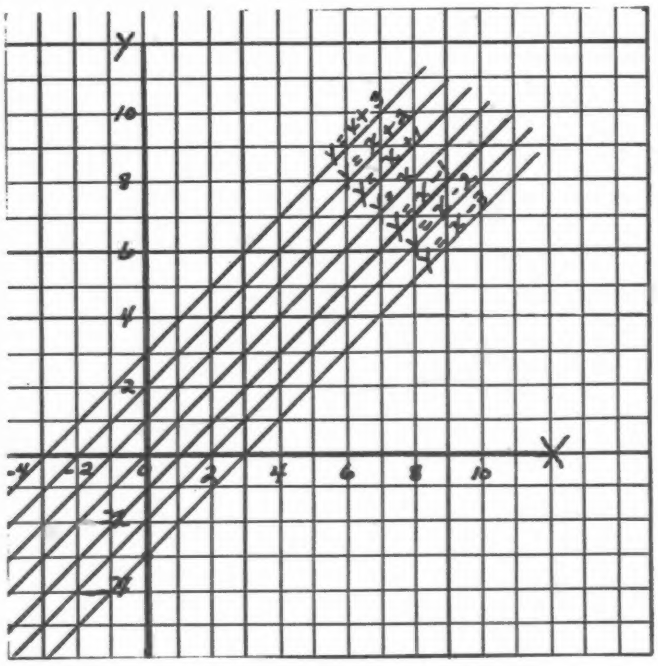
Pages 27-29 Exercises-5

1. Table A, formulas a and f * Table B, formulas c and g * Table C, formulas d and e * Table D, formulas b and k
2. a and f * c and g * d and e * b and h
3. Graph * All graphs rise to the right.



Exercises-5 (continued)

4a-c. Graph

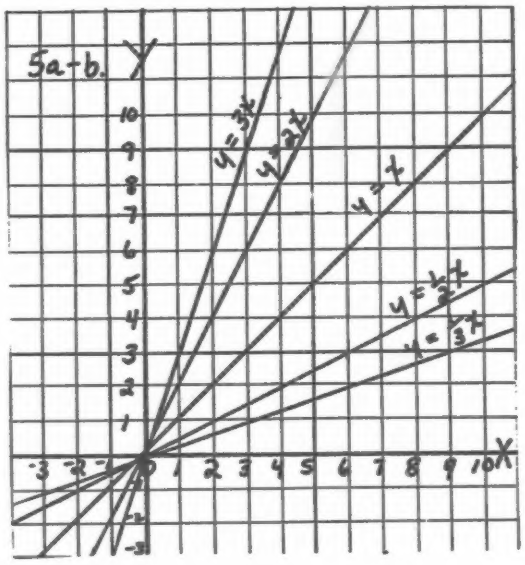


d. (0,100) * The same steepness

e. A line parallel to the others drawn in exercises 4a-c* (0,-100)

5a-b. Graphs

c. A line through the origin which rises very rapidly * A line through the origin which rises very slowly

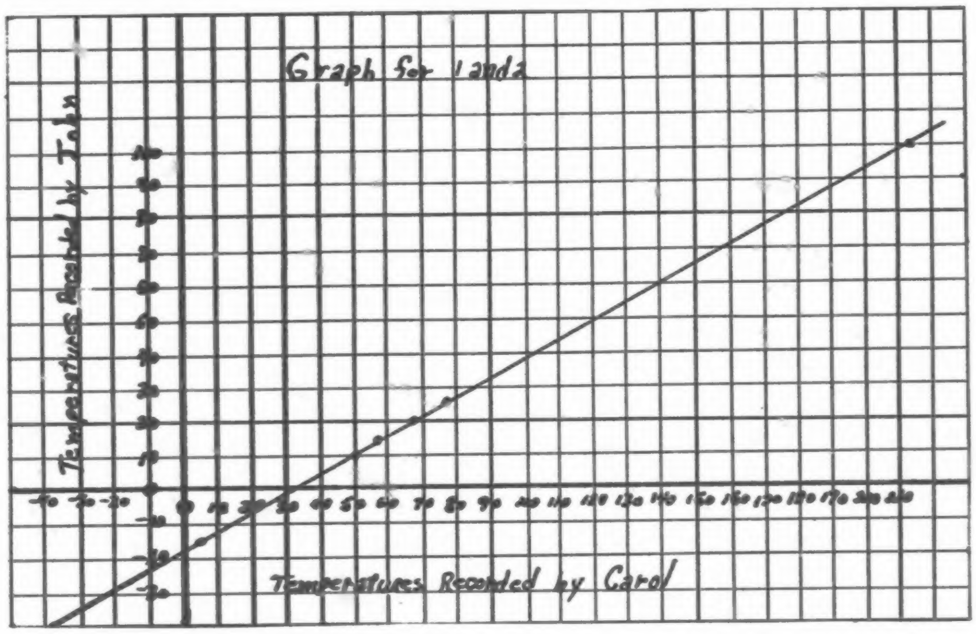


6a. Answers vary. One possibility is $y = 3x + 2$.

b. Answers vary. One possibility is $x + y = 3$.

Page 30 Class Discussion-6

1. Graph
2. Yes * Graph



3. 0° * 26° * -23° (Answers are approximate.)
4. 50° * 172° * -22° (Answers are approximate.)

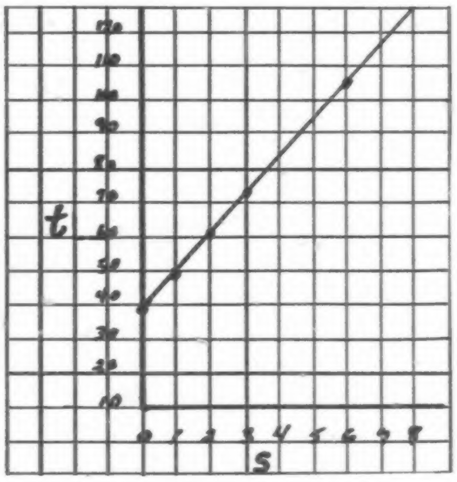
Pages 31-35 Exercises-6

1a. 95°F. * 88°F. * 100°F. (Answers are approximate.)
 b. 95°F. * 87-4/5°F. * 100-2/5°F.
 c. Using the formula is more accurate.
 d. Using the graph is faster.
2. The same table of values fits both formulas.
 a. $C = 5/9\ (27)$. * $C = 15$.
 b. 15°C. * The same
3a. 5°C. * $C = 5/9\ (F-32)$.
 b. 140°F. * $F = 9/5C + 32$.
 c. 80°C. * $C = 5/9\ (F-32)$.
4a.

s	1	2	3	4	5	6
t	50	61	72	83	94	105

Exercises-6 (continued)

4b. Graph



c.

s	$1\frac{1}{2}$	$4\frac{1}{2}$	$3\frac{1}{2}$	$5\frac{1}{2}$	0
t	55	$88\frac{1}{2}$	77	100	39

(Answers are approximate.)

5.

x	30	50	20	0	-20	-35	-40	-10
y	50	70	40	20	0	-15	-20	10

6a. $y = x$.
b. $y = 7 - x$.
c. $y = 2x - 4$.
7. $y = x + 20$.

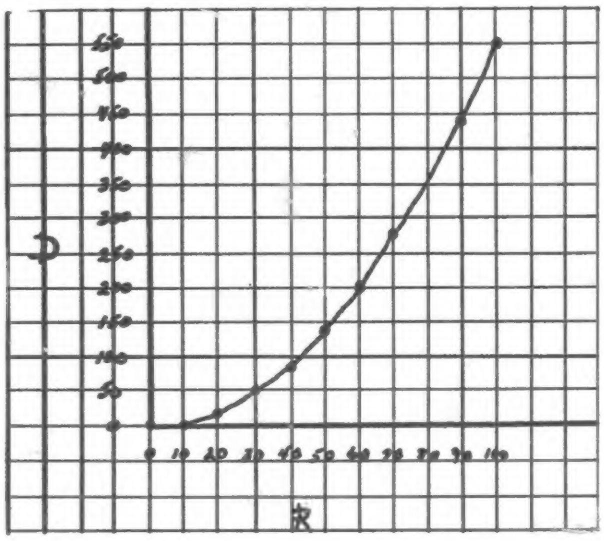
Page 36 Class Discussion-7

1. 352 feet * Answers vary.
2a. a x a * 10 x 10 * R x R x R
b. 100 * 400 * 900 * 1600
3.

R	0	10	20	30	40	50	60	70	90	100
D	0	5.5	22	49.5	88	137.5	198	269.5	445.5	550

Class Discussion-7 (continued)

4. Graph



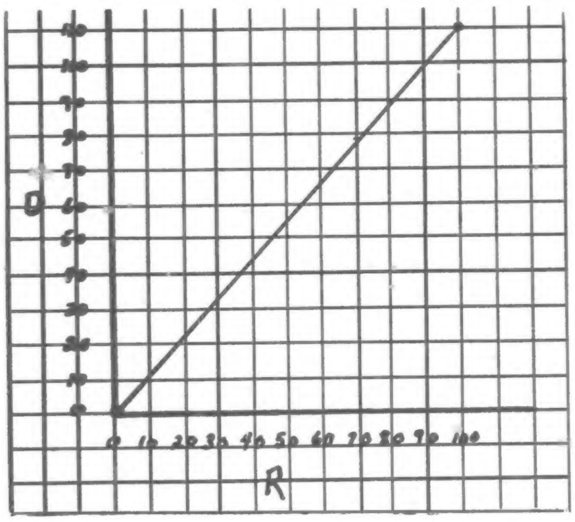
5. See graph in exercise 4. * It is a curve.
6. About 350 feet * Approximately the same
7a. About 67 feet * About 167 feet
b. About 73 miles per hour * About 42 miles per hour
c. He measured the length of the skid marks for each driver. Using the formula $D = .055R^2$, he calculated the stopping distance for a car traveling at the speed limit. If the skid marks were greater in length than the stopping distance he calculated, he would conclude that the driver was speeding.

Pages 38-44 Exercises-7

1a.

R	0	10	20	30	50	100
D	0	11	22	33	55	110

b. Graph



Exercises-7 (continued)

1c. The graph of $D = 1.1R$ is a straight line. The other graph was a curve.

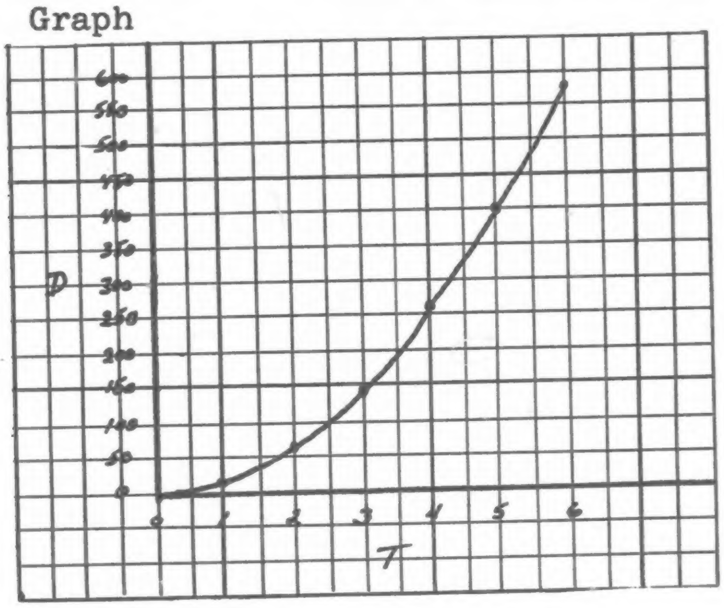
d. About 173 feet * About 288 feet * About 601 feet

e. About 74 feet

2a.

T	0	1	2	3	4	5	6
D	0	16	64	144	256	400	576

b. Graph



c. 100 feet

d. About 5.6 seconds

3a.

x	0	1	2	3	4	5
y	5	6	7	8	9	10

b.

x	0	1	2	3	4	5
y	2	4	6	8	10	12

Exercises-7 (continued)

3c.

x	0	1	2	3	4	5
y	12	11	10	9	8	7

d.

x	0	1	2	3	4	5
y	0	2	8	18	32	50

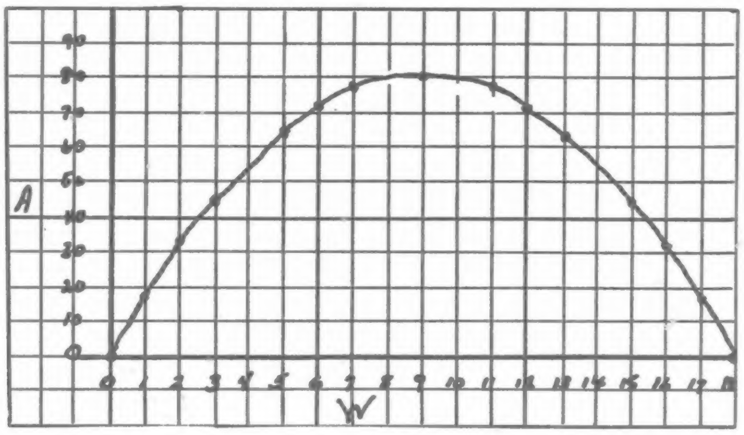
e.

x	0	1	2	3	4	5
y	4	5	8	13	20	29

4a. The graphs for formulas a, b, d, e
b. Formula c
c. Formulas a, b, c
5a. 17 ft. * 17 sq. ft.
b. 16 ft. * 32 sq. ft.
c. 45 sq. ft.
d.

W	1	2	3	5	6	7	9	12	14	16	17	18
L	17	16	15	13	12	11	9	6	4	2	1	0
A	17	32	45	65	72	77	81	72	56	32	17	0

e. Graph



Exercises-7 (continued)

5f. 81 sq. ft.
6a. For example, (5,26), (6,37), (7,50), (8,65) *
For (x,y): $y = x^2 + 1$.
b. For example, (12,2), (24,1), (4,6), ($\frac{1}{2}$, 48) *
For (x,y): $xy = 24$.

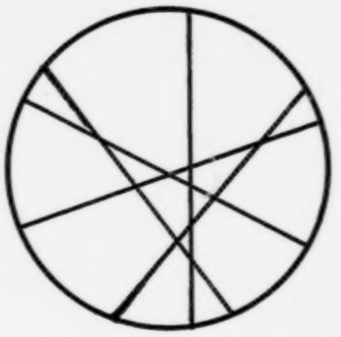
Pages 42-44 Class Discussion-8

1a. 1
b. 2
c. 4
d. 6
e. 8
f. For c not zero, $p = 2c$, where p is the number of pieces and c is the number of cuts.
(The student may say that after the first cut, the number of pieces increases by two with each cut.)
g. 16
2. Yes. The third cut should not pass through the intersection of the first two cuts.

3. 11
4.

Number of Cuts	0	1	2	3	4
Greatest Number of Pieces Possible	1	2	4	7	11

5a. 16
b.

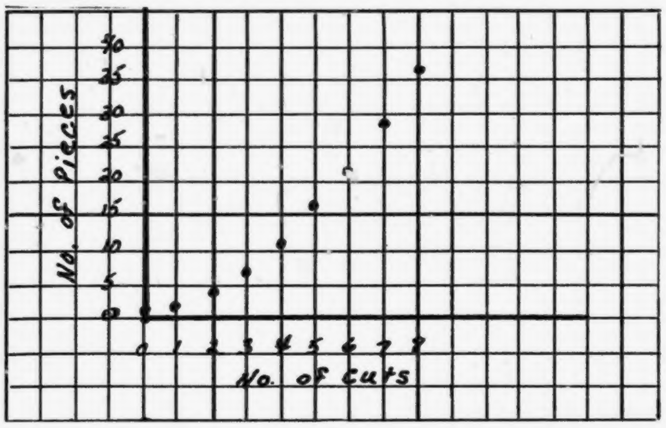


Class Discussion-8 (continued)

6.

Number of Cuts	0	1	2	3	4	5	6	7	8
Greatest Number of Pieces Possible	1	2	4	7	11	16	22	29	37

7. Graph * All of the solutions



8. No. It is meaningless to talk about a fractional number of cuts or a fractional number of pieces of pie. Some points on the graph do not represent solutions of the problem.
9. No
10. The graph for the stopping distance was a continuous line.

Pages 44-46 Exercises-8

1. Yes. * Both C and C + 1 are whole numbers and one of them is even. So either C or C + 1 is divisible by 2. Either $\frac{C}{2}$ or $\left(\frac{C + 1}{2}\right)$ is a whole number. Thus the product $\frac{C(C + 1)}{2}$ is a whole number because the product of two whole numbers is a whole number. Then $\frac{C(C + 1)}{2} + 1$ is a whole number because the sum of a whole number and 1 is a whole number.
   a. 11
   b. 37
   c. 92

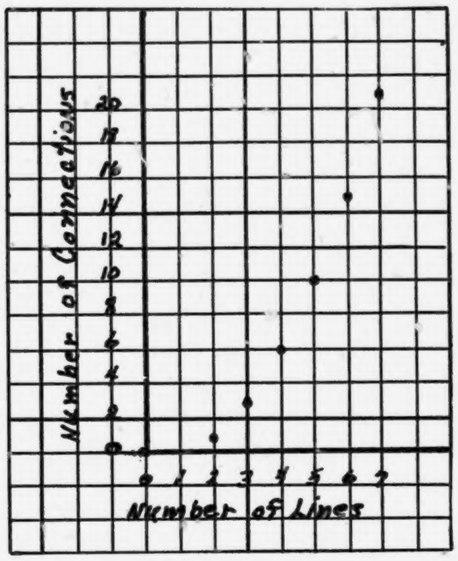
Exercises-8 (continued)

2a.

No. of Lines	2	3	4	5	6	7
No. of Connections	1	3	6	10	15	21

b. Graph

c. No



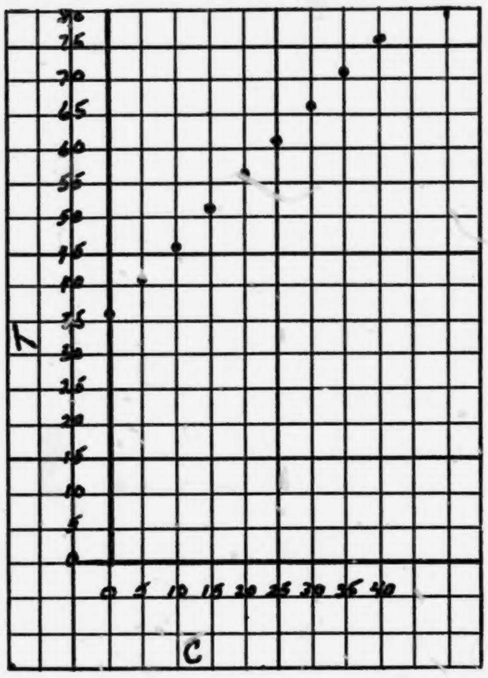
3a. 47°F.

b.

C	0	5	10	15	20	25	30	35	40
T	37	42	47	52	57	62	67	72	77

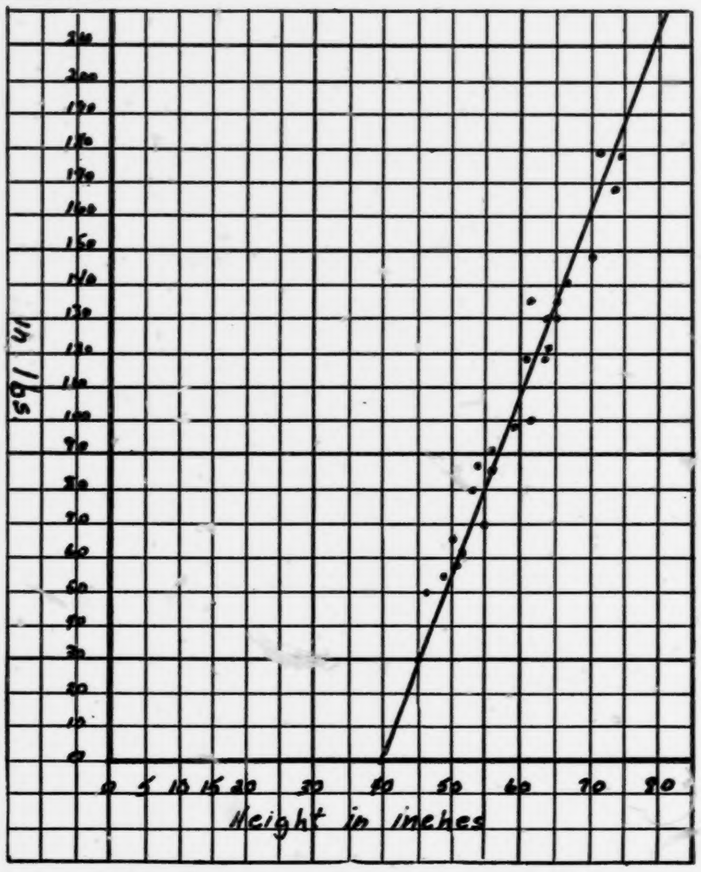
c. Graph

d. No * A cricket cannot make a fraction of a chirp.



Page 46 Class Discussion-9

1. Graph
2. They are on a straight narrow band.
3. Graph
4a. Closely
 b. Answers vary.
5a. 110 lb.
 b. Answers vary.
 c. Approximately the same
 d. 60 lb. * Answers vary.
 e. 160 lb. * Answers vary.
6a. 0 lb.
 b. About 0 lb.
7a. 92 inches (7 ft. 8 in.)
 b. Answers vary.
8a. No * The formula applies only within a limited range of heights and to persons of average build.
 b. No * See answer to 8a.
 c. About 50 in. to 80 in.



(Graph for #1 and 3)

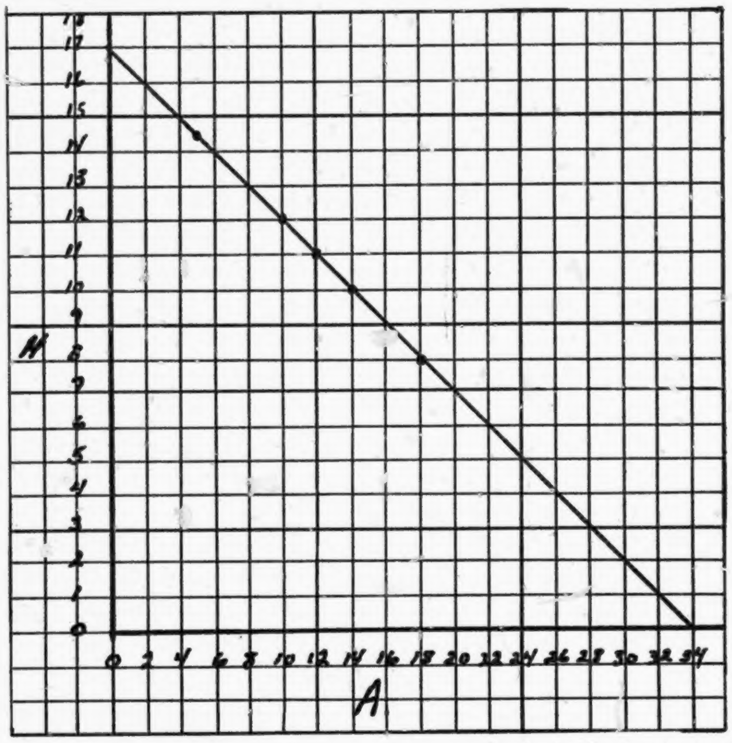
Pages 49-51 Exercises-9

1a.

A	5	10	12	14	18
H	$14\frac{1}{2}$	12	11	10	8

Exercises-9 (continued)

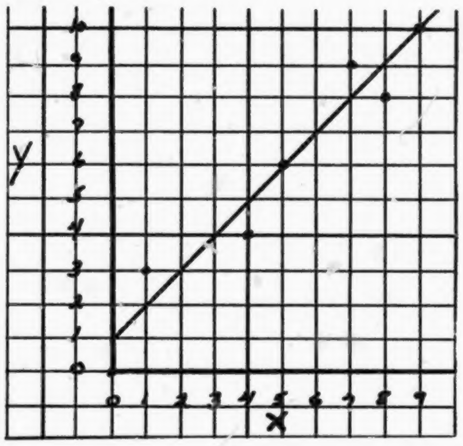
1b. Graph



c. Falls to the right
d. 16-1/2 hr.
e. 0 hr. * 0 hr.
f. No * 1 yr. to 18 yr.

2a-b. Graph
c. $y = x + 1$.
d. No

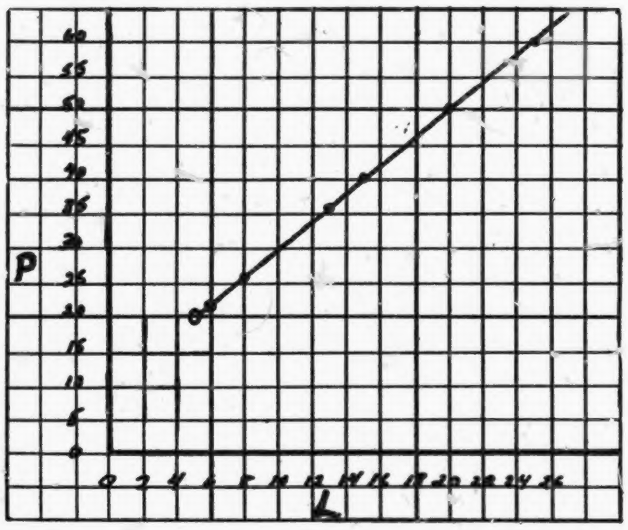
3a. An infinite number
b. 40 in.
c. 50 in.
d. $P = 2L + 10$.
e.



L	6	8	13	15	20	25
P	22	26	36	40	50	60

Exercises-9 (continued)

3f. Graph



4. No * No
(The legs are not long enough to form a triangle.)

Pages 52-54 Class Discussion-10

1. A greater weight is needed.
2. The distance must be cut in half.
3. 3 in.
4. 24 oz.
5. WD = 48.
6a. 9.6 in.
b. 4.8 in.
c. Yes
7a. 9.6 oz.
b. 4.8 oz.
8. 48 in.
9. C

Pages 55-57 Exercises-10

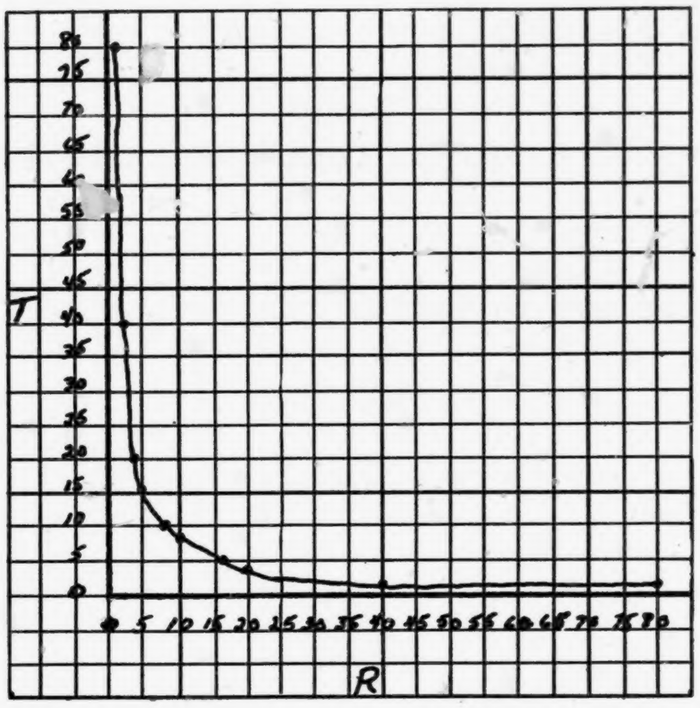
1. The formula is appropriate because 3 x 240 = 720. This is the number of pounds of weight times the number of feet in the distance from the fulcrum.
a. 6 ft. * 4 ft. * 7.2 ft.
b. 12 ft. * 14.4 ft.
c. No. He would need to sit 18 ft. from the fulcrum and the teeter-totter has only 15 ft. on each side of the fulcrum.

Exercises-10 (continued)

1d. 8 ft. from the fulcrum
e. 3-1/3 ft.
2a. 100 lb.
b. 222-2/9 lb.
3a. 2 hr.
b. 2-2/3 hr.
c. 4 hr.
d.

R	80	40	20	16	10	5	4	2	1
T	1	2	4	5	8	16	20	40	80

e. Graph



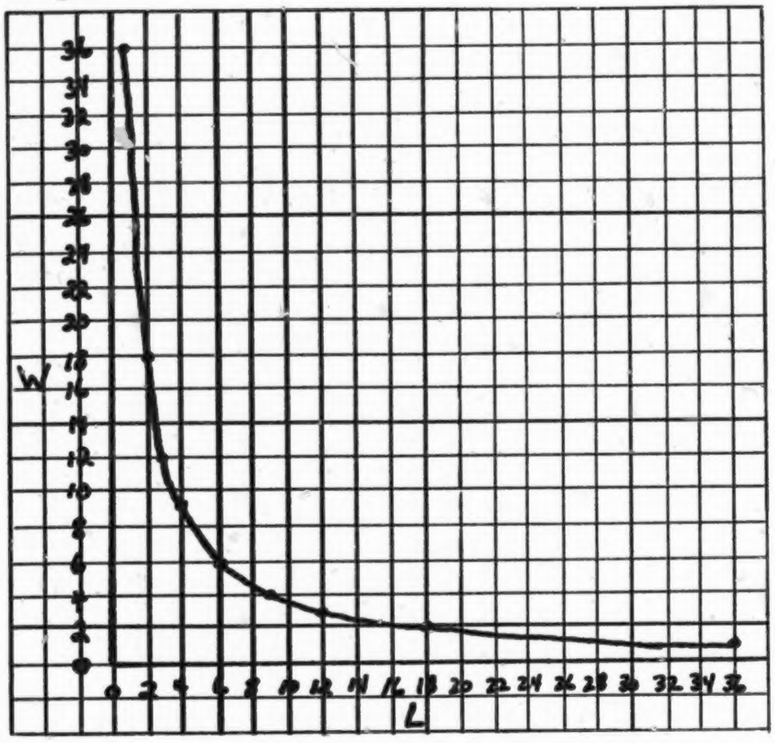
4a. For example: (6,6), (4,9), (3,12), (1,36), (12,3), (9,4)
b.

L	36	72	144	288
W	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

L	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
W	36	72	144	288

Exercises-10 (continued)

4c. Graph



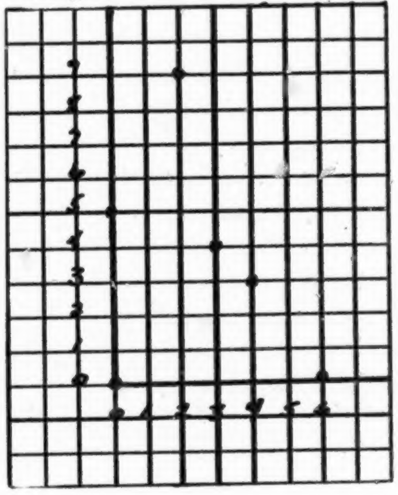
d. No * No * If L or W were zero, then the product LW would be zero, not 36.

5a. 6 ft. by 6 ft.

b. No * By decreasing the width, we can always decrease the length. Thus the perimeter can be increased indefinitely.

Pages 57-60 Review Exercises

1. Graph



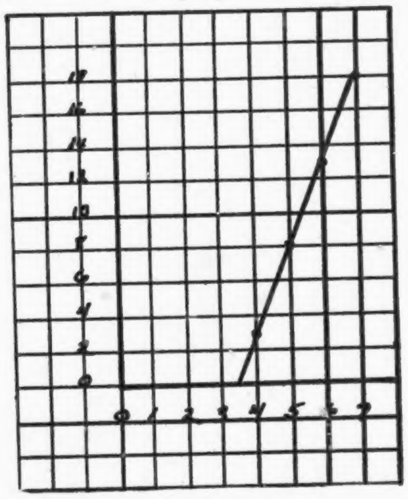
Review Exercises (continued)

2.

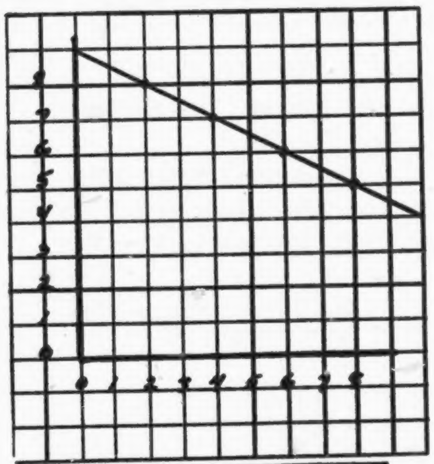
x	0	2	4	6	8	10
y	10	8	6	4	2	0

3. Any 3 ordered pairs of the form (a, a - 1)

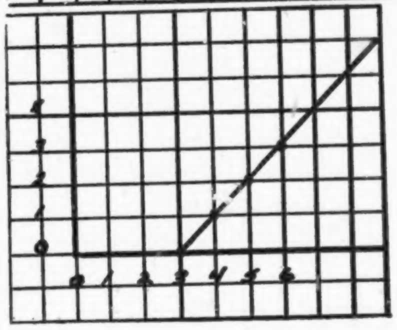
4a. Graph * Pattern: As x increases by 1, y increases by 5 * Any five pairs of the form (a, 5a - 17)



b. Graph * Pattern: As x increases by 1, y decreases by 1 * Any five pairs of the form (a, $9 - \frac{1}{2}a$)

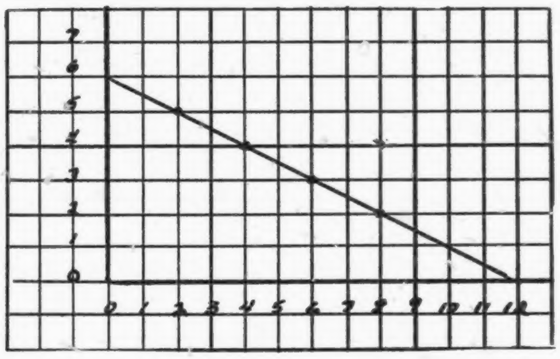


c. Graph * Pattern: As x decreases by 1, y decreases by 1 * Any five pairs of the form (a, a - 3)



Review Exercises (continued)

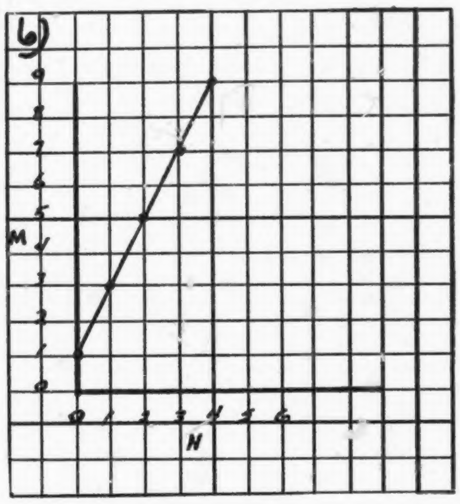
4d. Graph * Pattern: As x decreases by 2, y increases by 1 * Any five pairs of the form $(a, 6 - \frac{1}{2}a)$



5. Table A, formula d * Table B, formula b * Table C, formula a

6.

N	0	1	2	3
M	1	3	5	7



7a.

x	1	2	3	4	5	6
y	7	14	21	28	35	42

b.

x	3	6	9	12	15	18
y	1	2	3	4	5	6

Review Exercises (continued)

7c.

x	1	2	3	4	6	8	12
y	24	12	8	6	4	3	2

d.

x	0	1	2	3	4	5	6
y	0	1	4	9	16	25	36

e.

x	2	4	6	8	10
y	8	6	4	2	0

8a. c, d
b. c, e
c. a, b, d
9. $y = \frac{1}{3}x + 4$

10a.

x	2	9	4	10
y	7	21	11	23

b.

x	2	9	4	10
y	33	54	39	57

c.

x	2	9	4	10
y	0	49	14	56

d.

x	2	9	4	10
y	20	405	80	500

Review Exercises (continued)

10e.

x	2	9	4	10
y	10	87	22	106

f.

x	2	9	4	10
y	$11\frac{2}{3}$	14	$12\frac{1}{3}$	$14\frac{1}{3}$

g.

x	2	9	4	10
y	$\frac{1}{2}$	$5\frac{3}{4}$	2	$6\frac{1}{2}$

UNIT 2 - PROPERTIES OF OPERATIONS WITH NUMBERS

Answer Key

Pages 2-3 Class Discussion-1

1a. 3 * 5
b. 6
c. Saturday
2. 7
3. Wednesday
4. 24
5. 21
6. 7
7. It is three times the number of days in a week.
8. Find the sum of today's number and the number of days after today. Subtract the greatest multiple of seven which is not more than the sum. The resulting number corresponds to the day of the week that is in question.
9. 28
10. Wednesday

Page 3 Exercises-1

1.

Day	Sun.	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
Number	0	1	2	3	4	5	6

2a. $2 + 30 = 32$. $32 - 28 = 4$. 4 corresponds to Thursday.
b. Tuesday
c. Wednesday
d. Sunday
e. Tuesday
f. Tuesday
3. Saturday ($3 + 31 = 34$. $34 - 28 = 6$.)
4. Wednesday
5. Thursday

Pages 4-5 Class Discussion-2

1. 2
2. 2
3. $5 \oplus 4 = 2$.

Class Discussion-2 (continued)

4a. 3
b. 5
c. 2
d. 4
5. Whenever the result of ordinary addition is less than seven
6. 6
7. $5 \otimes 4 = 6$.
8. Find the result of ordinary multiplication and subtract the greatest multiple of 7 that is not greater than the product.
9. 1

Pages 5-6 Exercises-2

1a. 4
b. 0
c. 2
d. 5
e. 3
f. 2
g. 5
h. 1
i. 1
j. 4

2.

⊕	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

3.

⊗	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Pages 7-9 Class Discussion-3

1. Yes
2. Yes
3. Yes
4. 30 is the product of 5 and 6 obtained by ordinary multiplication, not by circle multiplication. $5 \otimes 6 = 2$, and 2 is a number in the original set.
5. No
6. No
7. 1 and 3 (Answers vary.)
8. 2 and 3 (Answers vary.)

Page 9 Exercises-3

1. Yes
2. Yes
3. Yes
4. No
5. Yes
6. Yes
7. Yes
8. Yes * Yes
9. Yes * Yes

Pages 10-11 Class Discussion-4

1. Yes * Yes
2. 1 * 1 * Yes
3. Yes
4. Start at 0 on the number circle and move 5 spaces 3 times in the direction of the arrows. * Yes
5. b, e

Pages 11-13 Exercises-4

1a. 20
b. 10
c. 15
d. 30
e. 24
2. No
3. Yes
4. Yes
5a. 647
b. 647
c. 675
d. 675
6. They are the same. * They are the same.
7. Commutative property of addition of whole numbers
8a. 2,736
b. 4,320
c. 4,320
d. 2,736

Exercises-4 (continued)

9. a and d; b and c * Commutative property of multiplication of whole numbers
10. They are the same. * Commutative property of circle addition
11. They are the same. * Commutative property of circle multiplication
12. (There are many possible answers.) (1) Fold the paper along the diagonal as in exercise 10, and record the number which coincides with each vacant square. (2) As you travel across a row or down a column, the numbers appear in consecutive order as they do on the number circle.
13. Fold the paper along the diagonal as before and record the number which coincides with each vacant square.
14. b ◬ a = c.

Page 15 Class Discussion-5

1a. 1
b. 1
2. Yes
3a. 6 and 6; yes
b. 1 and 1; yes
4. Yes
5. Either (4 + 5) + 3 or 4 + (5 + 3)

Pages 15-18 Exercises-5

1. 4
2. 4 * Yes
3. Yes (5)
4. Yes (6)
5. Yes (6)
6. Yes
7. 4
8. 4
9. 4
10. Changing the grouping of four numbers does not change the result of the circle addition.
11. Yes
12. 4 * 4 * Yes * Yes
13. Yes
a. 3
b. 3

Exercises-5 (continued)

14. Yes * No
15. Yes * Answers vary. Some possibilities are:
$4 + 3 = 3 + 4 = 7$; $4 \times 3 = 3 \times 4 = 12$.
16. 7 * 26 * 30
17. 30 * 15 * 29
18. A binary operation is an operation defined for two numbers.
a. No
b. No
19a. 5
b. 7
20. No
21a. 2
b. 8
22. No

Pages 19-20 Class Discussion-6

1. 1
2. 1
3. Yes
4. Yes
5. $4 \otimes (5 \oplus 3) = 4 \otimes 1 = 4$;
$(4 \otimes 5) \oplus (4 \otimes 3) = 6 \oplus 5 = 4$.
6. $(5 \otimes 2) \oplus (5 \otimes 4)$
7. $5 \otimes (2 \oplus 4) = 5 \otimes 6 = 2$;
$(5 \otimes 2) \oplus (5 \otimes 4) = 3 \oplus 6 = 2$.
8. (1) First find the sum of 2 and 3. Then find the product of 4 and that sum.
(2) First find the products of 4 and 2 and of 4 and 3. Then find the sum of those products.

Pages 20-21 Exercises-6

1. 1
2. $6 \otimes (5 \oplus 6) = 6 \otimes 4 = 3$;
$(6 \otimes 5) \oplus (6 \otimes 6) = 2 \oplus 1 = 3$.
3. $5 \otimes (6 \oplus 4) = 5 \otimes 3 = 1$;
$(5 \otimes 6) \oplus (5 \otimes 4) = 2 \oplus 6 = 1$.
4. No
5. 2
6. 0
7. No
8. No
9. No
10. 120

Exercises-6 (continued)

11. Yes
12. 12 x (10 + 9) = 12 x 19 = 228;
(12 x 10) + (12 x 9) = 120 + 108 = 228. * Yes

Pages 22-23 Class Discussion-7

1. Yes
2. Yes
3. Yes
4a. 10 x (8 + 9) = (10 x 8) + (10 x 9).
b. 50 x (12 + 8) = (50 x 12) + (50 x 8).
5a. 25 x (7 + 12) = (25 x 7) + (25 x 12).
b. 35 x (9 + 2) = (35 x 9) + (35 x 2).
6. Multiply the number of rows by the number of x's in one row, or multiply the number of columns by the number of x's in one column. (Multiply 3 x 7.)
7a. See exercise 6. (Multiply 3 x 4.)
b. See exercise 6. (Multiply 3 x 3.)
8. Because the two parts shown in exercise 7 were obtained by separating the chart shown in exercise 6 * Yes
9a. 5 x 41 = (5 x 40) + (5 x 1) = 200 + 5 = 205.
b. 192
c. 252
d. 294

Pages 23-24 Exercises-7

1a. 55
b. 544
c. 264
d. 496
e. 231
f. 1,200
2a. 6(40 + 5) = (6·40) + (6 · 5) = 240 + 30 = 270
b. 117
c. 208
d. 213
e. 301
f. 256
3a. 92
b. 215
c. 245
d. 243
e. 204
f. 335

Pages 26-28 Class Discussion-8

1. 0
2. Yes * Yes * Yes * Yes
3. Yes * Because of the commutative property of addition, a + 0 = 0 + a.
4. No * Yes

Class Discussion-8 (continued)

5. 1
6. 25 * 40 * Yes
7. No * Yes
8. For any whole number a, it is true that $a \times 1 = a$.
9. Addition of a whole number and 1 results in the next higher whole number.
10. Yes * 0 * Yes * For any whole number a, it is true that $a \cdot 0 = 0$. * Yes
11. No * No * (We assume here that we are working only with positive whole numbers and that the student is not familiar with negative numbers.)
12. No * No
13. Yes
14. No

Pages 28-29 Exercises-8

1a. 1
b. 2
c. 3
d. 4
e. 5
f. 6
g. 0

2a. 6
b. 3
c. 2
d. 5
e. 4
f. 1

3. 5 * 4 * 3 * 2
4. 0
5. Yes
6. 1 * 4 * 2 * 6
7. No
8. Yes
9. No * Yes (zero)
10. No * Yes (one)

Page 30 Class Discussion-9

1. 2 * Yes
2. Yes * 4
3. 5

4a. $2 \ominus 6 = 3$ because $3 \oplus 6 = 2$.
b. $3 \ominus 4 = 6$ because $6 \oplus 4 = 3$.
c. $2 \ominus 3 = 6$ because $6 \oplus 3 = 2$.
d. $0 \ominus 6 = 1$ because $1 \oplus 6 = 0$.
e. $4 \ominus 6 = 5$ because $5 \oplus 6 = 4$.

5. Yes

Page 31 Exercises-9

1a. 5
b. 5
c. 2
d. 4
e. 4
f. 1
g. 4
h. 3
i. 3
j. 6
k. 1
l. 3
2. 4
3. 4
4.

Number	0	1	2	3	4	5	6
Additive Inverse	0	6	5	4	3	2	1

5. 3 * 5
6. Instead of subtracting, add the first number and the additive inverse of the second number.
7a. $3 \ominus 5 = 3 \oplus 2 = 5$.
b. $2 \ominus 4 = 2 \oplus 3 = 5$.
c. $0 \ominus 5 = 0 \oplus 2 = 2$.
d. $3 \ominus 6 = 3 \oplus 1 = 4$.
e. $1 \ominus 4 = 1 \oplus 3 = 4$.
f. $5 \ominus 4 = 5 \oplus 3 = 1$.
g. $0 \ominus 3 = 0 \oplus 4 = 4$.
h. $0 \ominus 4 = 0 \oplus 3 = 3$.
i. $1 \ominus 5 = 1 \oplus 2 = 3$.
j. $5 \ominus 6 = 5 \oplus 1 = 6$.
k. $0 \ominus 6 = 0 \oplus 1 = 1$.
l. $2 \ominus 6 = 2 \oplus 1 = 3$.

Pages 32-33 Class Discussion-10

1. We know that 27 ÷ 3 = 9 because 9 x 3 = 27.
2. Yes (four)
3. 9 * Yes
4. No whole number * No
5. No
6. 0 ÷ 6 = 0 because 0 x 6 = 0.
7. Yes * 0 ÷ 100 = 0 because 0 x 100 = 0.
8. Yes
9a. No * Because the product of the quotient (if there is one) and the divisor must equal the dividend * No * No * Because C x 0 = 0 for any whole number C.
b. An infinite number * Yes
10. Yes
11. Yes (three) * Yes
12. 6
13. 6 * Yes * No number
14. Yes

Page 34 Exercises-10

1. 5 ÷ 3 = 4, or 5 ÷ 4 = 3.
2a. 6
b. 5
c. 2
d. 5
e. 3
f. 5
g. 4
h. 2
i. 3
j. 5
3a. 5 ⨸ 2 = 6.
b. 3 ⨸ 2 = 5.
c. 1 ⨸ 4 = 2.
d. 6 ⨸ 4 = 5.
e. 2 ⨸ 3 = 3.
f. 4 ⨸ 5 = 5.
g. 3 ⨸ 6 = 4.
h. 3 ⨸ 5 = 2.
i. 1 ⨸ 5 = 3.
j. 1 ⨸ 3 = 5.
4. Yes
5. 1 and 1, 2 and 4, 3 and 5, 6 and 6 * Yes (1 is, and 6 is.)
6. Yes. Instead of dividing, multiply the dividend by the multiplicative inverse of the divisor.
7a. $5 \otimes 4 = 6$
b. $6 \otimes 3 = 4$
c. $4 \otimes 3 = 5$
d. $2 \otimes 6 = 5$
e. $1 \otimes 2 = 2$
f. $1 \otimes 3 = 3$
g. $5 \otimes 5 = 4$
h. $6 \otimes 2 = 5$
i. $1 \otimes 4 = 4$
j. $5 \otimes 2 = 3$

Page 36 Review Exercises

1. See answers to exercises 2 and 3, pages 5-6.
2. Commutative property of circle addition
3. Commutative property of circle multiplication
4. $(3 + 4) + 6 = 7 + 6 = 13$;
$3 + (4 + 6) = 3 + 10 = 13$.
5. Associative property of addition in ordinary arithmetic
6. Yes
7. $6 \times (3 + 5) = 6 \times 8 = 48$;
$(6 \times 3) + (6 \times 5) = 18 + 30 = 48$. * Distributive property of multiplication over addition in ordinary arithmetic
8a. Distributive property of multiplication over addition
b. Commutative property of addition
c. Commutative property of multiplication
d. Associative property of addition
e. Commutative property of multiplication
f. Associative property of multiplication
9a. $9 \cdot 8 = 8 \cdot \underline{9}$ * Commutative property of multiplication
b. $4 + 5 = \underline{5} + 4$ * Commutative property of addition
c. $5(4 + 9) = \underline{20} + \underline{45}$ * Distributive property for multiplication over addition
d. $(4 + 5) + 8 = 4 + (\underline{5} + \underline{8})$ * Associative property of addition

Review Exercises (continued)

9e. $(20 \cdot 5) \cdot 6 = (\underline{5} \cdot 20) \cdot \underline{6}$ * Commutative property of multiplication

10a. $2 \cdot (6 + 3) = (2 \cdot 6) + (2 \cdot 3)$.
b. $3 \cdot (7 + 1) = (3 \cdot 7) + (3 \cdot 1)$.
c. $5 \cdot (2 + 4) = (5 \cdot 2) + (5 \cdot 4)$.
d. $4 \cdot (8 + 3) = (4 \cdot 8) + (4 \cdot 3)$.
e. $2\ (4 + 9) = (2 \cdot 4) + (2 \cdot 9)$.
f. $8\ (3 + 7) = (8 \cdot 3) + (8 \cdot 7)$.
g. $8\ (2 + 6) = (8 \cdot 2) + (8 \cdot 6)$.
h. $7\ (3 + 4) = (7 \cdot 3) + (7 \cdot 4)$.
i. $(6 \cdot 3) + (6 \cdot 9) = 6\ (3 + 9)$.
j. $(9 \cdot 2) + (9 \cdot 8) = 9\ (2 + 8)$.

11a. $3 \cdot (2 + 4) = 3 \cdot 6 = 18$;
$(3 \cdot 2) + (3 \cdot 4) = 6 + 12 = 18$.
b. $6 \cdot (7 + 1) = 6 \cdot 8 = 48$;
$(6 \cdot 7) + (6 \cdot 1) = 42 + 6 = 48$.
c. $4 \cdot (3 + 5) = 4 \cdot 8 = 32$;
$(3 \cdot 4) + (5 \cdot 4) = 12 + 20 = 32$.
d. $(5 \cdot 6) + (5 \cdot 4) = 30 + 20 = 50$;
$5\ (6 + 4) = 5 \cdot 10 = 50$.

12a. 5
b. 5
c. 3
d. 3
e. 3
f. 6
g. 6
h. 2
i. 4
j. 4

13. 0

14.

Number	0	1	2	3	4	5	6
Additive Inverse	0	6	5	4	3	2	1

15.

Number	1	2	3	4	5	6
Multiplicative Inverse	1	4	5	2	3	6

16. 0 * 1

UNIT 3
Answer Key

Page 2 Class Discussion-1

1a. True
b. Open
c. False
d. Open
e. True
f. True
2a. 3
b. 9
c. 3
d. 2
e. 15
f. 3
3a. 3 + 8 = 11 is true.
b. 9 - 3 = 6 is true.
c. 2(3) + 4 = 10 is true.
d. 3(2) - 1 = 5 is true.
e. 15 - 9 = 6 is true.
f. 3(3) - 4 = 5 is true.
4. Sentence c

Pages 3-4 Exercises-1

1. Answers vary.
2a. False
b. True
c. Open
d. Open
3a. 11
b. 5
c. 21
d. 5
e. 7
f. 8
g. 7
h. 7
i. 7
j. 8
4a. 3(6) - 5 = 13. * True
b. 7(10) + 3 = 80. * False
c. 5(3) - 8 = 2. * False
d. 4(7) + 17 = 45. * True
e. 7(5) + 12 = 54. * False
5a. 7
b. 5
c. 7
d. 3
e. 9
f. 6
g. 8
h. 13
i. 24
j. 6
6a. 5
b. 3
c. 8
d. 2
e. 2

Pages 4-5 Class Discussion-2a

1a. Mary opened the door.
b. Mary moved the chair two feet backward.
c. Mary took the eraser from Phil and put it in its original position.
2a. Closing the book
b. Walking two steps forward
c. Putting the shoe on
d. None possible

Class Discussion-2a (continued)

2e. (Some students may argue that mussing your hair would undo the combing. Others may argue that, since the hair cannot be mussed in exactly the same way and the original situation cannot be restored, the combing action cannot be undone.)

f. None possible.
(An undoing action takes a situation which has been effected by one action and restores the original situation. In a physical situation, an action can never quite be reversed. For example, after Phil writes on the board and Mary erases it, the chalk is no longer quite the same length, there is probably a smudge on the blackboard, and the eraser is dirty. On the other hand, in abstract situations, such as those which occur in operations with numbers, undoing may be complete.)

3a. Closing the book and returning it to its original place on the shelf
b. Removing the pencil from the pair of compasses and putting it down in its original position
c. Replacing the sheet of paper in the notebook and closing the notebook
d. Replacing the piece of chalk in its original position, walking to your original seat, and sitting down

4a. Subtracting 2 from the result
b. Dividing the result by 5
c. Multiplying the result by 3
d. Adding 7 to the result

Page 7 Class Discussion-2b

1a. Multiplication and addition
b. Addition and multiplication

2.

Instructions For Doing	Open Phrase	Instructions For Undoing
Take a number.	x	Divide by five.
Multiply by five.	$5x$	Subtract three.
Add three.	$5x + 3$	Take the open phrase.

Class Discussion-2b (continued)

3a. Take a number.
Multiply by two.
Subtract seven.
b. Take a number.
Multiply by four.
Add two.
c. Take a number.
Multiply by 5.
Subtract 4.

Pages 8-9 Exercises-2

1a. Returning home from school
b. Going to sleep
c. None possible
d. Turning off the TV set
e. Forgetting the poem
2. x
$4x$
$4x + 5$

3a.

	INSTRUCTION	OPEN PHRASE
	Take a number.	x
	Multiply by five.	$5x$
	Add ten.	$5x + 10$
b.	Take a number.	x
	Divide by two.	$\frac{x}{2}$
	Subtract seven.	$\frac{x}{2} - 7$
c.	Take a number.	x
	Multiply by six.	$6x$
	Add four.	$6x + 4$

4a.

	INSTRUCTION	OPEN PHRASE
	Take a number.	x
	Multiply by seven.	$7x$
	Add five.	$7x + 5$
b.	Take a number.	x
	Divide by six.	$\frac{x}{6}$
	Add four.	$\frac{x}{6} + 4$
c.	Take a number.	x
	Multiply by five.	$5x$
	Add three.	$5x + 3$

Exercises-2 (continued)

5a. Take a number.
Multiply by three.
Subtract eight.

b. Take a number.
Multiply by two.
Add fourteen.

c. Take a number.
Subtract five.

d. Take a number.
Multiply by five.
Add one.

e. Take a number.
Multiply by four.

6.

INSTRUCTION	OPEN PHRASE
Take the open phrase.	$7x - 3$
Add three.	$7x$
Divide by seven.	x

7a.

	INSTRUCTION	OPEN PHRASE
	Take the open phrase.	$2x - 9$
	Add nine.	$2x$
	Divide by two.	x
b.	Take the open phrase.	$x + 4$
	Subtract four.	x
c.	Take the open phrase.	$3y + 5$
	Subtract five.	$3y$
	Divide by three.	y
d.	Take the open phrase.	$\frac{r}{2} + 10$
	Subtract ten.	$\frac{r}{2}$
	Multiply by two.	r

8.

	INSTRUCTIONS FOR DOING	OPEN PHRASE	INSTRUCTIONS FOR UNDOING
a.	Take a number.	x	Divide by five.
	Multiply by five.	$5x$	Subtract three.
	Add three.	$5x + 3$	Take the open phrase.
b.	Take a number.	x	Divide by two.
	Multiply by two.	$2x$	Add one.
	Subtract one.	$2x - 1$	Take the open phrase.

Exercises-2 (continued)

	INSTRUCTIONS FOR DOING	OPEN PHRASE	INSTRUCTIONS FOR UNDOING
8c.	Take a number.	x	Divide by four.
	Multiply by four.	$4x$	Subtract seven.
	Add seven.	$4x + 7$	Take the open phrase.
d.	Take a number.	x	Multiply by seven.
	Divide by seven.	$\frac{7}{x}$	Subtract five.
	Add five.	$\frac{7}{x} + 5$	Take the open phrase.

9a. $x + 7$
b. $2x - 3$
c. $3x + 5$
d. $x + 2x$
e. $5 + x$
f. $4x - 6$
g. $x + 2x + 4$
h. $7x - 4$

Pages 10-11 Class Discussion-3

1a. 16 years old
b. 6
c. 7 cents
2a. 3
b. 7
c. 6
d. 7
3b. $2x - 5 = 9$.
c. $2x = 5 + 7$.
d. $2x - 6 = 8$.
4. Answers vary. Some possibilities are:
  a. Three times my age, when added to 61, results in 100.
  b. If there were 3 less than 5 times the number of children in my family, there would be exactly one dozen.
  c. If you divide the number of years I have lived in my home by four, the result is 2.

Pages 11-13 Exercises-3

1a. 5
b. 15
c. 7
d. 25
e. 32
f. 10

Exercises-3 (continued)

2a. 2
b. 7
c. 5
d. 9
e. 3
f. 11
g. 6

3a. If a number is multiplied by two and seven is added to the product, the result is fifteen.
b. If a number is divided by three and four is added to the quotient, the result is eight.
c. If a number is multiplied by five and four is added to the product, the result is nineteen.
d. If a number is multiplied by seven and three is subtracted from the product, the result is eighteen.

4a. 4
b. 12
c. 3
d. 3

5. Answers vary.
6. Answers vary.

7a. 12
b. 7
c. 16
d. 8
e. 12
f. 44
g. 25
h. 16
i. 15
j. 4

8b. $x + 3x = 28$.
c. $2x + 13 = 45$.
d. $4x - 5 = 27$.
e. $\frac{x}{2} + 3 = 9$.
f. $\frac{x}{4} = 11$.
g. $\frac{x}{5} - 3 = 2$.
h. $\frac{x}{4} + 3 = 7$.
i. $\frac{x}{3} + 4 = 9$.
j. $7x - 6 = 22$.

9a. Twice a number minus three is eleven.
b. If a number is divided by four and five is added to the quotient, the result is eight.
c. Five times a number plus two times the same number is twenty-eight.
d. If a number is divided by two and seven is subtracted from the quotient, the result is ten.
e. If four is subtracted from the sum of a number and twice the number, the result is five.
f. If a number is divided by three and five is added to the quotient, the result is nine.
g. The sum of twice a number and eleven is seventeen.

10a. 7
b. 12
c. 4
d. 34
e. 3
f. 12
g. 3

11a. $3x + 3 = 21$
b. $100 - 2x = 12$, or $2x + 12 = 100$.
c. $p + 7 = 23$, or $4s + 7 = 23$.

Pages 15-16 Class Discussion-4a

1. One * One
2. Yes * If x is replaced by any whole number the resulting sentence will be true.
3. The solution set has an infinite number of members.
4. 3 and 4
5. No * The x^2 becomes larger much faster than $7x$. Thus their difference keeps increasing in size.
6. Two
7. No
8. No replacement for x will make the sentence true.
9. None
10. No * It may have no members, a finite number of members, or an infinite number of members.

Page 17 Exercises-4a

1a. { }
b. {3}
c. { }
d. {2,3}
e. {0,1,2,...}
f. {4,5}
g. {10}
h. {6}
i. { }
(If the student has worked with negative numbers, he may argue that {-6} is the solution set. In this unit we are using only whole number replacements for the variable.)
j. {0}
k. { }
l. {0}
m. {0,1,2,...}

2. a, c, i, k
3. e, m

Pages 18-20 Class Discussion-4b

1a. No * No
b. (4)
c. All except (4)
d. $x + 4 = 6$. * Yes * Add 6 to each side. * Add 5 to each side. * Add 9 to each side.
e. Add 100 to each side. * Add 99 to each side. * Add 94 to each side. * Add 101 to each side.
f. Yes * Answers vary. A partial answer is as follows. The "equal" sign means that the expressions on the left and right sides represent the same number. When a number is added to two different expressions for the same number, the resulting expressions should be equal. Thus if a number is a solution

1f. for one open sentence, it should also be a solution for a second open sentence obtained by adding the same number to each side of the first open sentence. (A complete answer would include the reverse process where the first open sentence is obtained from the second by subtracting the same number from each side.)

g. (1) $2x + 3 = 9$.
   (2) $2x + 4 = 10$.
   (3) $2x + 5 = 11$.
   (4) $2x + 0 = 6$.

2a. All except (3)

b. (3)

c. $x + 11 = 20$. * Yes

d. Subtract 8 from each side. * Subtract 10 from each side.

e. Subtract 1 from each side. * Subtract 9 from each side. * Subtract 11 from each side.

f. Yes, if subtraction is possible * (See answer to exercise 1f.)

g. No * 4 * No * There is no whole number which can be added to 7 to obtain 4.

h. (1) $4x + 0 = 12$.
   (2) $4x + 4 = 16$.
   (3) $4x + 1 = 13$.

Pages 20-22 Class Discussion-4c

1. Addition Property of Equality for Open Sentences: If an open sentence expresses an equality and we add the same number to both sides, we get another open sentence that has the same solution set as the original open sentence.

2a. To undo subtraction of 5 from a number, we add 5.

b. $\{16\}$

c. $\{16\}$

d. The Addition Property of Equality for Open Sentences assures us that it is.

Page 23 Exercises-4c

1a. $t + 5 = 7$.
S_5 $t = 2$.
The solution set is $\{2\}$.

Exercises-4c (continued)

1b. {10}
c. {24}
d. {3}
e. {13}
f. {0}
g. {21}
h. {5}
i. {3}
j. {9}
k. {8}
l. {7}
2a. {26}
b. {20}
c. {11}
d. {11}
e. { }
f. { }
g. {4}
h. { }
i. {2}
j. {20}
3. e, f, h
4a. {10} * Subtraction property of equality for open sentences
b. {29} * Addition property of equality for open sentences
c. {0} * Subtraction property of equality for open sentences
d. {25} * Addition property of equality for open sentences
e. {18} * Subtraction property of equality for open sentences

Pages 24-26 Class Discussion-4d

1. x = 4. * {4} * Yes
2. 12x = 48. * 6x = 24. * 4x = 16. * 3x = 12. * 2x = 8. * The solution set of each is {4}.
3. No
4. Yes * Yes * The divisors must be whole numbers which can be divided evenly into the numbers on both sides of the open sentence.
5a. {8}
b. Yes
c. Multiply each side by 3. * Multiply each side by 9. * Multiply each side by 18.
6. Yes, except for multiplication by zero
7a. {8}
b. {0, 1, 2,...}
c. No

Pages 26-27 Class Discussion-4e

1. Division property of equality for open sentences
2a. 26
b. Multiplication property of equality for open sentences
c. Yes
d. Computation

Pages 27-28 Exercises-4e

1a. {14}
b. {5}
c. {40}
d. {63}
e. {0}
f. { }
g. {10}
h. {14}
i. {0}
j. {12}
k. {14}
l. { }.
2. f and l
3a. {12} * Division property of equality for open sentences
b. {24} * Addition property of equality for open sentences
c. {21} * Multiplication property of equality for open sentences
d. {16} * Subtraction property of equality for open sentences
e. {6} * Multiplication property of equality for open sentences and division property of equality for open sentences

Pages 28-30 Class Discussion-4f

1a. Yes * Yes
b. Yes
c. Subtraction property of equality for open sentences
d. Division property of equality for open sentences
2a. $x = 150$. * Yes * It has a variable on the left side and a whole number on the right side.
b. Yes
c. Addition property of equality for open sentences
d. Multiplication property of equality for open sentences
3. Answers vary. (The student may argue that if a first open sentence has the same solution as a second open sentence, and the second open sentence has the same solution set as a third, then the first open sentence has the same solution set as the first. If he argues in this way he has the correct idea. The mathematical idea which underlies the reasoning is the transitive property of the equality relation.)

Page 31a Class Discussion-4f (continued)

4a. Add 15 to each side. * Addition property of equality for open sentences

Class Discussion-4f (continued)

4b. Add 5 to each side. * Addition property of equality for open sentences

c. Subtract 2 from each side. * Subtraction property of equality for open sentences

d. Subtract 5 from each side. * Subtraction property of equality for open sentences

e. Add 18 to each side. * Addition property of equality for open sentences

5a. None needed

b. Divide both sides by 3. * Division property of equality for open sentences

c. Divide both sides by 7. * Division property of equality for open sentences

d. Multiply both sides by 3 or divide both sides by 2. * Multiplication property or division property of equality for open sentences

e. Divide both sides by 5. * Division property of equality for open sentences

6.

$$\frac{3x}{4} - 12 = 9.$$

A_{12} $$\frac{3x}{4} = 21.$$

M_4 $$3x = 84.$$

D_3 $$x = 28.$$

The solution set is $\{28\}$.

Page 30 Exercises-4f

1a. $\{5\}$
b. $\{9\}$
2a. $\{4\}$
b. $\{7\}$
c. $\{19\}$
d. $\{9\}$
3a. $\{15\}$
b. $\{38\}$
c. $\{21\}$
c. $\{4\}$
d. $\{15\}$
e. $\{6\}$
f. $\{28\}$
g. $\{1\}$
h. $\{32\}$
d. $\{25\}$
e. $\{35\}$
f. $\{2\}$
e. $\{14\}$
f. $\{11\}$
i. $\{3\}$
j. $\{15\}$
k. $\{2\}$
l. $\{12\}$
g. $\{3\}$
h. $\{1\}$
i. $\{19\}$
j. $\{21\}$
g. $\{21\}$
h. $\{3\}$

Page 32 Class Discussion-5

1. $x + 4$
2. $4x + 8$

Class Discussion-5 (continued)

3. 28
4. $4x + 8 = 28$.
5. {5}
6. The width is 5 inches.

Page 32 Exercises-5

1. $3x + 7 = 46$. * {13} * The number is 13.
2. $4x + 14 = 54$. * {10} * The width is 10 in.
3. $4x - 5 = 39$. * {11} * The number is 11.
4. $2x + 8 = 42$. * {17} * Dick is 17 years old.
5. $2x + 29 = 735$. * {353} * Bob received 382 votes.
6. $2x + 800 = 12,800$. * {6000} * The empty truck weighs 6000 lb.
7. $3x + 11 = 53$. * {14} * The shortest side is 14 inches long.
8. $4x + 16 = 96$. * {20} * Jack invested $20.

Pages 34-35 Review Exercises

1a. Open
b. True
c. False
d. Open
e. Open
f. Open
g. Open
h. Open

2a. a,d,h
b. e
c. f,g
d. g

3a. Subtracting three from the result
b. Replacing the newspaper in its original position and closing the door
c. Dividing the result by seven
d. Erasing the underlining and closing the book

4a. $3x + 4$
b. $\frac{x}{5} - 7$
c. $7x - 3$

5a. $3x - 5$
b. $3x + 7$
c. $\frac{x}{2} + 6$

6a. {13}
b. {2}
c. {34}
d. {3}
e. {15}
f. {8}
g. {6}
h. {25}
i. {2}
j. {7}

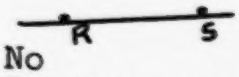
7a. $4x = 80$. * {20} * The number is 20.
b. $4x = 56$. * {14} * The shorter piece is 14 feet long.
c. $2x + 50 = 602$. * {276} * Joan got 276 votes.
d. $10(2x) + 5x = 75$, or $25x = 75$. * {3} * Bill has 6 dimes.

Class Discussion-3 (continued)

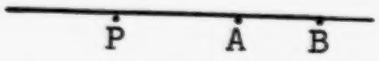
2a. No *

b. 0

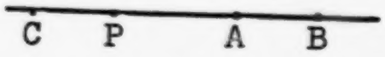
3a.



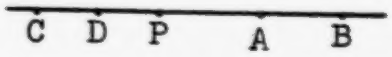
b. No
c. Exactly one
d. Yes - by any pair of points in the line
4b. An infinite number
c. An infinite number
e. Exactly one
f. Exactly one
5a. (Answers vary.)



b.

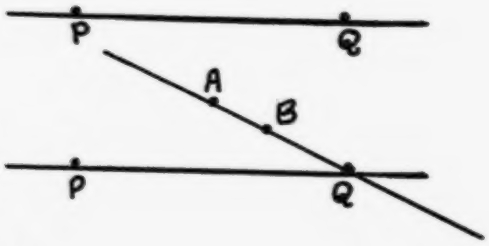


c.

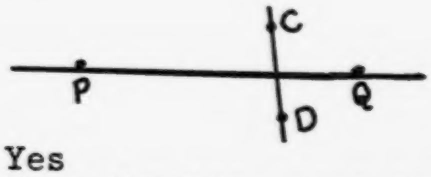


d. No
e. Same side
f. Two
6a.

b.



c. No
d.



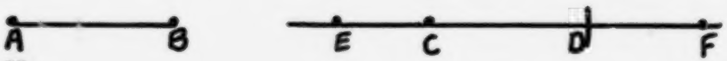
e. Yes
f. Draw the line containing the two points in question. The two given points are on opposite sides of line PQ if the line drawn meets line PQ in a point that is between the two given points. The two points

Class Discussion-3 (continued)

6f. are on the same side of line PQ if the line drawn does not meet line PQ in a point that is between the two given points.

7. (1) The set consisting of the points of the plane
   (2) The set consisting of all points in space on one side of the plane
   (3) The set consisting of all points in space on the other side of the plane

8a. Yes * They have the same size and shape.
b. Yes
c. Place the left end of the ruler at A and note the number that corresponds to point B. Then place the left end of the ruler at C and make a mark at a point in line EF that corresponds to the number obtained for B.
d.



e. No
f. If B is between A and C
9a. No
b. One
10a. No
b. No
c. No

Pages 15-17 Exercises-3

1. Segments VR, RS, ST, VT
2. Segments AB, BC, AC
3a. Sides VR and ST
b. Sides VT and RS
4. Sides VR and ST; Sides VT and RS
5. If C is between A and B
6a. AD and DC (Answers vary.)
b. AB and DC (Answers vary.)
c. AB and EH (Answers vary.)
7a. One
b. Three
c. Six

UNIT 4 - GEOMETRY

Answer Key

Page 1 Class Discussion-1

1. Taste, color, weight, food value
2. Size, shape, position in space

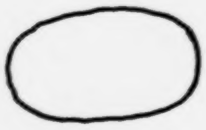
Pages 2-3 Exercises-1

1. No
2. Yes
3. No
4. Yes
5. No
6. Yes
7. No
8. Yes
9. No
10. Yes
11. Yes
12. Yes
13. Yes
14. No
15. Yes
16. Yes

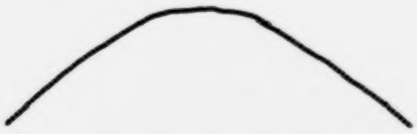
Pages 4-7 Class Discussion-2

1a. No
b. No
c. No
d. No

2a.

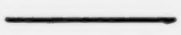


b.



c.

d.



e. c and d are straight; a and b are curved
3a, b, c. demonstrations
d. c is flat; a and b are not flat.
4a. B and C
b. C
c. Yes
d. Yes

Class Discussion-2 (continued)

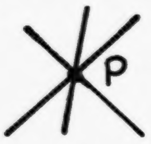
4e. An infinite number
f. Yes. They have no size. Between any two points there is always another point.
5a. No
b. If temperature is constant
6. A pencil line has thickness and endpoints.

Pages 7-8 Exercises-2

1. A grain of sand * A dot made with a finely sharpened pencil
2. The earth's diameter is so much smaller than the distance from earth to sun that for all practical purposes it can be neglected.
3a. Path of ray of light * Path of rifle bullet
b. A taut string stretched between two points
c. Single crease made by folding paper
d. Taut string stretched between 2 points
4a. No
b. Yes (if you neglect the thickness of the wire)
c. All points in a straight line containing the two points
5. Surface of a pond; surface of a window pane
6. Sheet of wallboard
7a. Steel tape measure (This assumes you are measuring along a straight line. A cloth tape measure would be better for measuring along a curve.)
b. The steel tape measure is more rigid - cloth tends to stretch.
8a. (1) Place them side by side so that one pair of ends is together.
(2) Measure them both with a ruler.
b. (1) The sticks retain their shape and size when moved.
(2) The shape and size of the ruler is not changed when it is moved.

Pages 9-15 Class Discussion-3

1a.



b. An infinite number
c. An infinite number

Exercises-3 (continued)

7d.

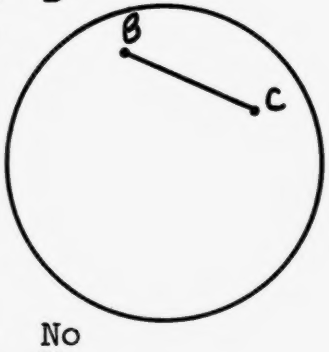
Points	Lines
2	1
3	3
4	6
5	10
6	15
7	21

e. 28
f. 66

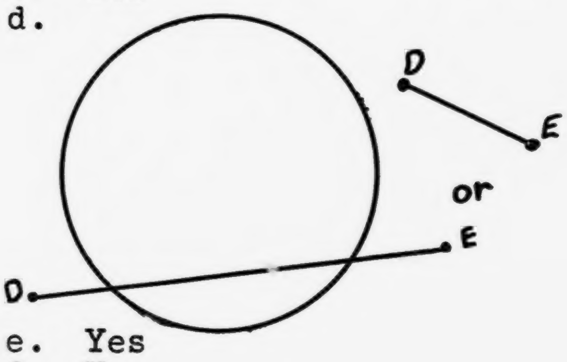
Pages 18-19 Class Discussion-4

1a. Distance between A and C
b. Distance between A and B
c. Distance between A and D
d. B
e. D

2a.



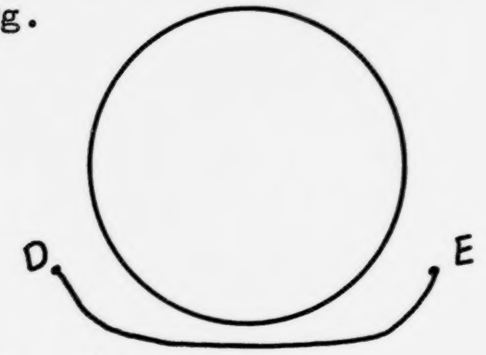
b. No
c. Yes
d.



e. Yes
f. Yes

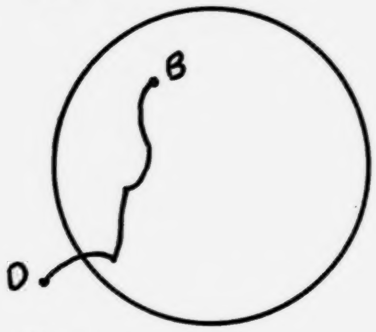
Class Discussion-4 (continued)

2g.



h. Yes

i.

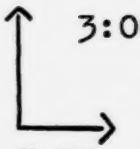


j. Yes
k. No
l. Yes
m. Yes
n. No
o. Yes
p. No
3a. Yes
b. No

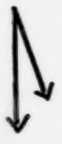
Pages 21-23 Class Discussion-5a

1. $\angle$ MNP, $\angle$ PNM, $\angle$ N
2. $\angle$ ABD, $\angle$ DBC, $\angle$ ABC

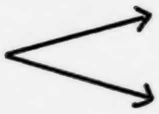
3a. 3:00



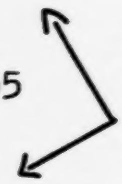
b. 5:30



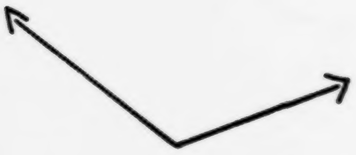
c. 4:10



d. 8:55

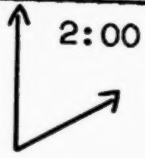


e. 2:50

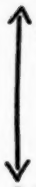


Page 21 Exercises-5a

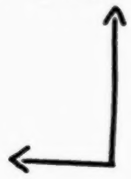
1a. (1) 2:00



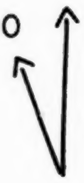
(2) 6:00



(3) 9:00



(4) 11:00



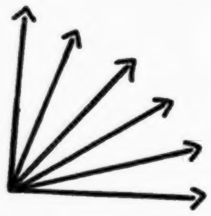
b. (1) Bandit at 3 o'clock
(2) Bandit at 7 o'clock
(3) Bandit at 2 o'clock
(4) Bandit at 10 o'clock

2.

	No. of rays	No. of angles formed
a.	2	1
b.	3	3
c.	4	6
	5	10

d. 15

e. 15



f. 36

3. $A = \frac{R \cdot (R-1)}{2}$

Pages 23-25 Class Discussion-5b

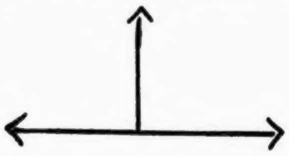
1a. Yes
b. No
c. Yes
d. No
e. Yes
2b. Yes
3b. Angle y
c. Angle z
d. No

f. Yes
g. No
h. Yes
i. No

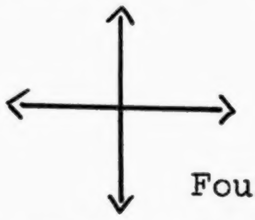
Class Discussion-5b (continued)

4. Angles x and y; Angles y and z; Angles w and z

5.



6.



Four right angles

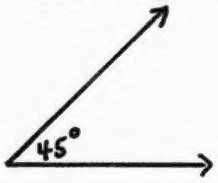
Pages 25-26 Exercises-5b

1. (Answers vary.) The corners where edges of the blackboard meet.

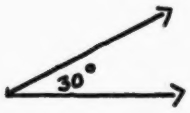
2. $\angle ABF$, $\angle IBL$, $\angle DBG$, $\angle JBM$, $\angle EBH$, $\angle KBN$, $\angle FBC$

Pages 26-29 Class Discussion-6a

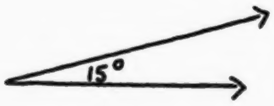
1a. 45°
 b. 30°
 c. Yes
3a. $\angle H$, $\angle K$
 b. $\angle B$, $\angle E$, $\angle N$
4a.



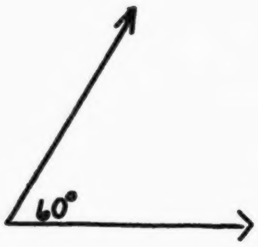
b.



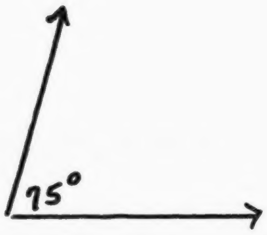
c.



d.

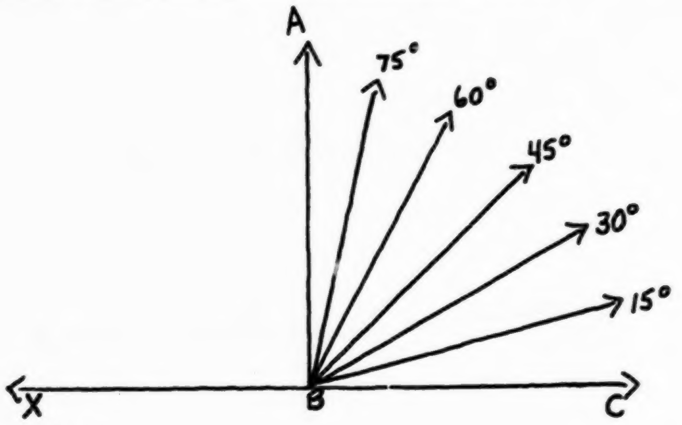


e.



Class Discussion-6a (continued)

5.



6. 180°
7. 180°
8. 140°; 120°; 90°; 105°; 107.5°

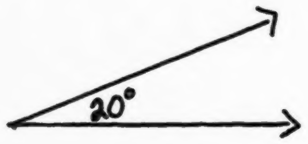
Pages 29-31 Exercises-6a

1a. 25°
b. 40°
c. 65°
d. 90°
e. 120°
f. 170°
g. 10°
h. 60°
i. 115°
j. 30°
k. 50°
l. 15°
m. 55°
n. 145°

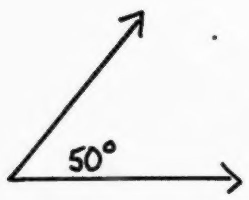
2a. Acute
b. Acute
c. Right
d. Obtuse
e. Obtuse
f. Acute
g. Right
h. Obtuse
i. Right

3a. 35°, 85°
b. Drawings vary.
c. Answers vary.

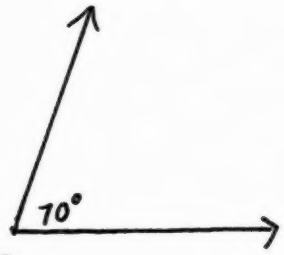
4a.



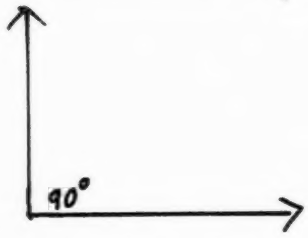
b.



c.

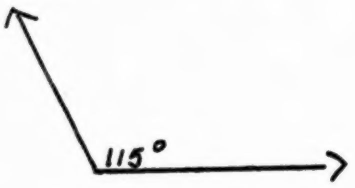


d.

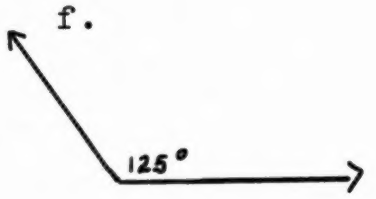


Exercises-6a (continued)

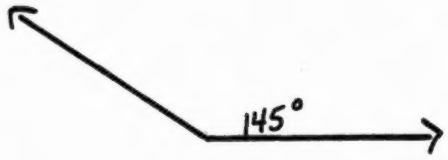
4e.



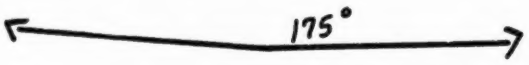
f.



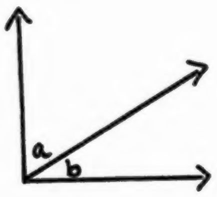
g.



h.

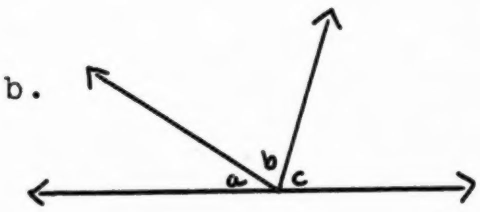


5a.



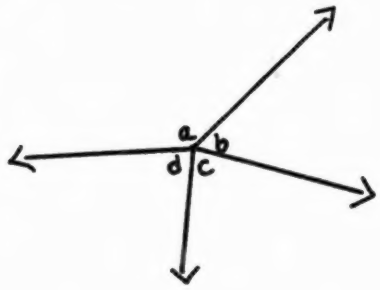
(Drawings vary.)

b.



(Drawings vary.)

c.



(Drawings vary.)

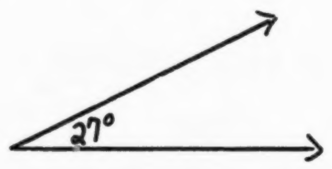
6a. 90°
b. 90°
c. 30°
d. 150°
e. 30°
f. 120°
g. 15°
h. 80°

Pages 31-33 Class Discussion-6b

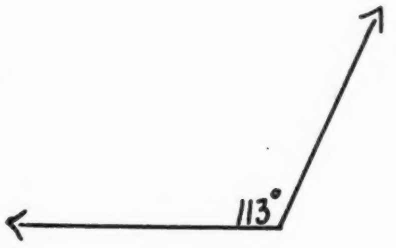
1a. 120°
b. 120°
2a. 100°
b. 100°
3a.

Inside No.	Outside No.
60	120
100	80
40	140
130	50
30	150
20	160
130	50

b. 170 * 110
4a. Lesser
b. Greater
5a. Acute
b. Obtuse
c. 180°
d. 45°
e. 135°
6a. About 65°
b. About 135°
c. About 90°
d. About 18°
e. About 125°
f. About 15°
7. See exercise 6.
8a.



b.

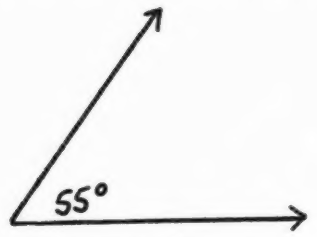


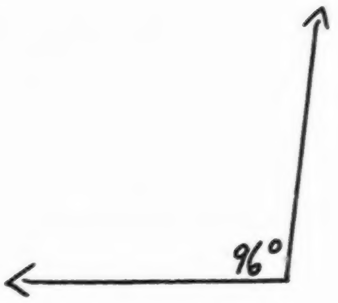
Pages 34-35 Exercises-6b

1a. 55° * 96°

Exercises-6b (continued)

1b.





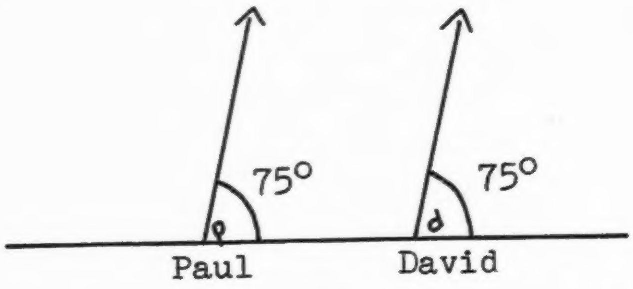
2a. <w = 95°
<x = 113°
<y = 58°
<z = 94°

b. Find the sum of the measures of the four angles. The sum of the measures of the four angles should be 360°.

3. Both $22\frac{1}{2}^\circ$ * Yes * Yes * Yes

4a. 67°

b.



5a. Angles n and s (Answers vary.)

b. Angles q and v,
Angles p and t,
Angles m and r

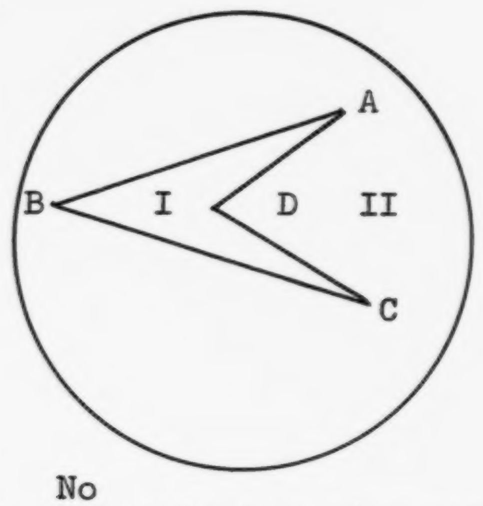
c. Yes

Page 37 Exercises-7

1. Segments MN, NO, OP, PM
2. Segments RS, ST, TU, UV, VW, WR
3. 3
4. 3 * 4 * 5 * The same number of sides as vertices
5. a, b, c, d

Exercises-7 (continued)

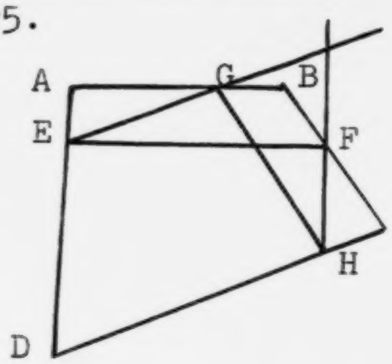
6.



7. No
8. The interior of a polygon can be completely included in the interior of a circle. The exterior cannot.
9. a, b, c, d
10. Not always
11. Yes
12. Yes
13. No
14. No
15. Yes
16. Point C * Yes * Angles ABC, BCD, CDE, DEA, EAB
17. Four
18a. Segments AB and DC; segments AD and BC
  b. Segments RS and TV; segments VR and ST
  c. Segments MN and OP; segments NO and PM
  d. Segments XY and ZW; segments WX and ZY
19. a, b, c
20. a, b
21. a
22. Yes * No
23. No * Yes
24a. The set of parallelograms is included in the set of quadrilaterals, but there are some quadrilaterals that are not in the set of parallelograms
  b. (1) All rectangles are quadrilaterals, but not all quadrilaterals are rectangles.
     (2) All squares are quadrilaterals, but not all quadrilaterals are squares.

Exercises-7 (continued)

25.



26. a, b, d
27. Yes
28. No
29. See drawings.

(29)

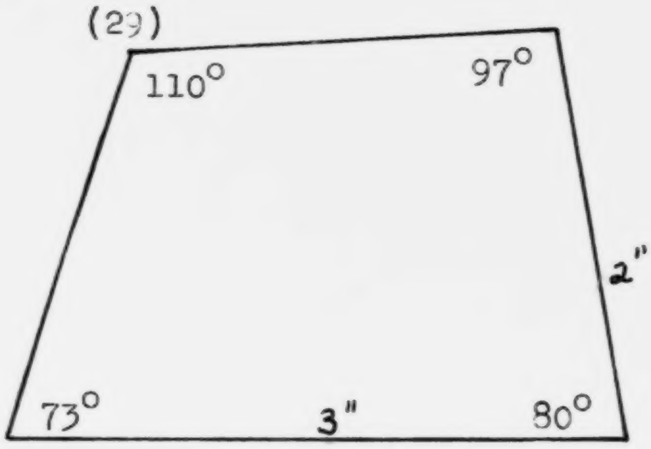

110°
97°
2"
73°
3"
80°

Figure 1

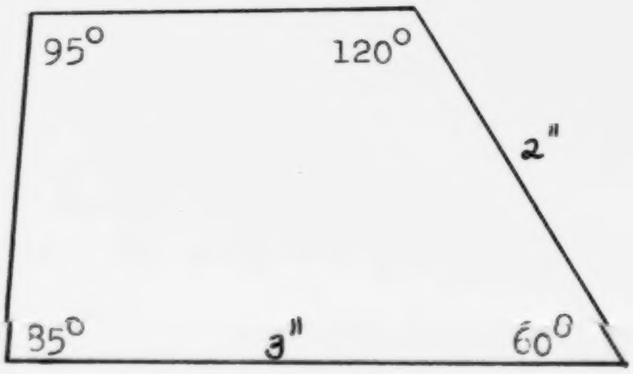

95°
120°
2"
85°
3"
60°

Figure 2

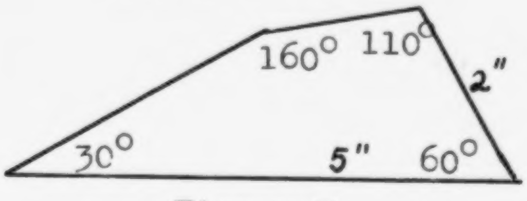

160°
110°
2"
30°
5"
60°

Figure 3

The second figure represents a trapezoid.

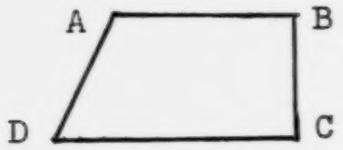
30a. Segments AB, CD, EF, AC, CE, BD, DF, AE, BF
b. Segments EC and CA;
Segments FD and DB
c. Lines EF, CD and AB
d. Line FB is perpendicular to lines EF, CD and AB.
e. Quadrilaterals EFDC, CDBA, EFBA

Exercises-7 (continued)

31a. AD and BC, AB and DC
  b. Yes
  c. Segments AB, BC, CD and DA; segments AE and EC; segments ED and EB

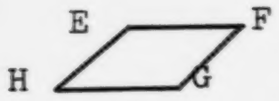
32a. All angles must be right angles.
  b. All sides must be the same length.

33a.



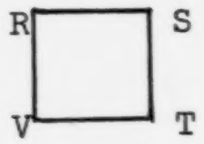
* No

b.



* No

c.

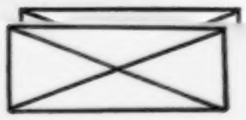


* A square

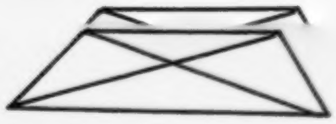
Pages 43-44 Class Discussion-8

1. Diagonal DB

2.



OR



3a.

Number of Sides	No. of diagonals from one vertex
3	0
4	1
5	2
6	3
7	4
8	5
9	6
10	7

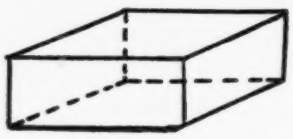
Class Discussion-8 (continued)

3b. 47
c. D = S - 3

Page 45 Class Discussion-9

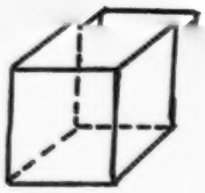
1. Triangle DCE; triangle ADE; triangle ABE
2. Segments BC, CD, DA, ED, EA, EB
3. A, B, C, D, E
4. a, b, c * w, x, y
5. (w) cylinder;
   (x) cone;
   (y) sphere
6. Angles ABC, BCD, CDA, DAB, EAD, ADE, DEA, EDC, DEC, ECD, BCE, CEB, EBC, ABE, AEB, EAB

7a. b
b.



c. 6
d. 12
e. 8

8a.



b. 6
c. 12
d. 8

9a. 8
b. 14
c. 8

Page 47 Exercises-9

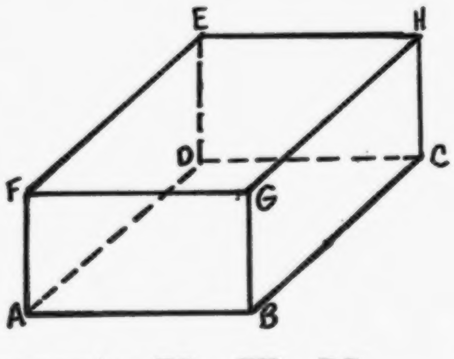
1.

Figure	F = Number of Faces	V = Number of Vertices	E = Number of Edges
a	5	5	8
b	6	8	12
c	5	6	9
d	8	8	14
e	7	10	15
f	6	6	10
g	11	18	27
h	10	10	18
i	9	9	16
j	62	60	120
k	8	12	18
l	8	6	12
m	Answers vary.		

m. The sum of the number of faces and the number of vertices is two more than the number of edges.

n. $E = F + V - 2$

2a.



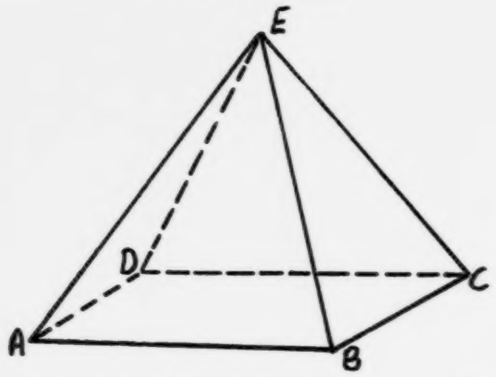
b. Segments FG, EH, DC

c. Segments GH, EF, ED, CH

d. The faces bounded by quadrilaterals ABCD and EFGH (Answers vary.)

Exercises-9 (continued)

3a.

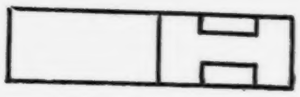


b. Segments AD and BC, or AB and DC

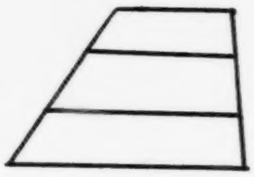
c. None

4. b * c * a

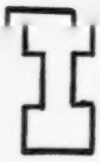
5.



Top view



Right side view

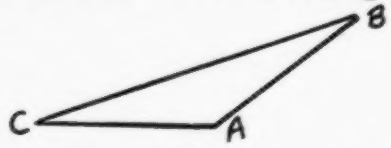


Back view

Pages 51-53 Class Discussion-10

1. An angle of a triangle is an angle which includes two sides of the triangle, and whose vertex is a vertex of the triangle.

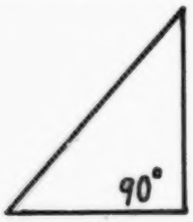
2.



3. No * The sides would not meet.

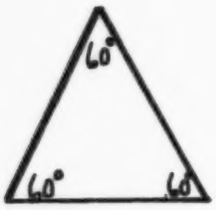
Class Discussion-10 (continued)

4.

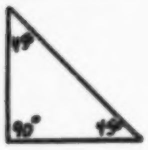


5. No * The second and third sides would not meet.

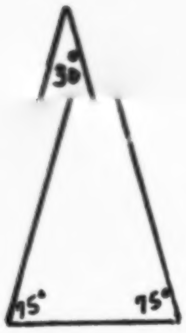
6a.



b.



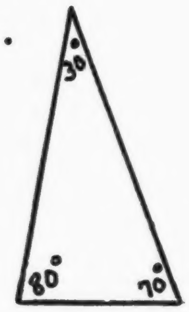
c.



d. Impossible

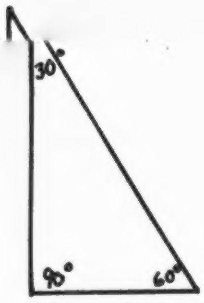
e. Impossible

f.



g. Impossible

h.

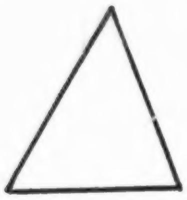


7. No * a, b, c, f, h

8. 180°

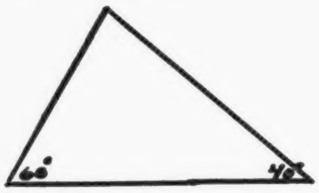
9. d. 150°
   e. 210°
   g. 160°

10.



* 180°

Class Discussion-10 (continued)

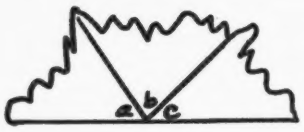
11. 

a. 80°
b. 80°

12. No

13. No * No

Page 54 Exercises-10

1. 

Angles a, b, and c represent the three angles of the original triangle.

2.

	m<A	m<B	m<C
Example	70°	25°	85°
Triangle I	47°	39°	94°
Triangle II	23°	65°	92°
Triangle III	20°	59°	101°
Triangle IV	165°	7°	8°
Triangle V	80°	$50\frac{1}{2}$°	$49\frac{1}{2}$°

3a. 47°
b. 122°
c. $58\frac{1}{2}$°
d. $24\frac{1}{2}$°

4a. 69°
b. 62°
c. 28°
d. 41°

Exercises-10 (continued)

5.

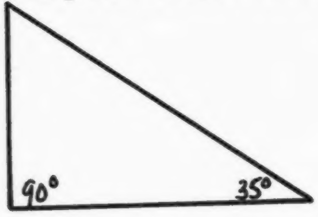
	m<BAC	m<B + m<C
a.	50°	130°
b.	57°	123°
c.	115°	65°
d.	90°	90°

6. n°

7a. 23°

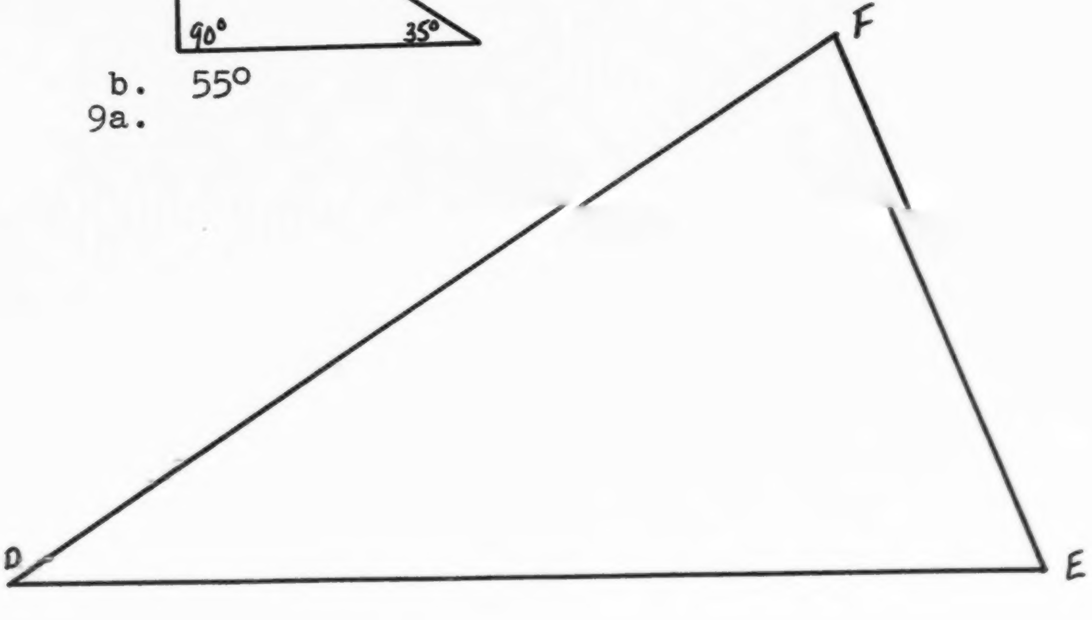
b. Impossible situation.

8a.



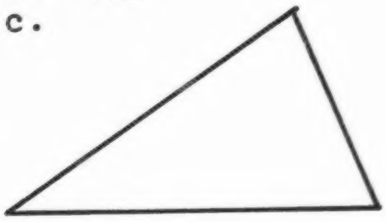
b. 55°

9a.



b. Yes

c.



Exercises-10 (continued)

9d. An infinite number

10a. $m<A = 119^\circ$,
$m<B = 88^\circ$,
$m<C = 79^\circ$,
$m<D = 74^\circ$

b. 360°

11. Yes

11a. Two

b. 180°

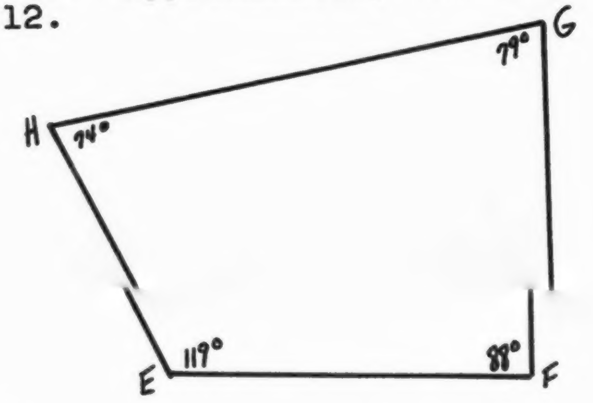
c. 180°

d. 360°

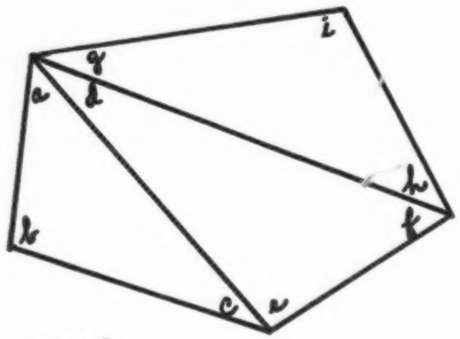
e. Yes

f. Approximately the same

12.



13.



$m<a + m<b + m<c = 180^\circ$.

$m<d + m<e + m<f = 180^\circ$.

$m<g + m<h + m<i = 180^\circ$.

$(m<a + m<b + m<c) +$
$(m<d + m<e + m<f) +$
$(m<g + m<h + m<i) = 540^\circ$.

14. 720°

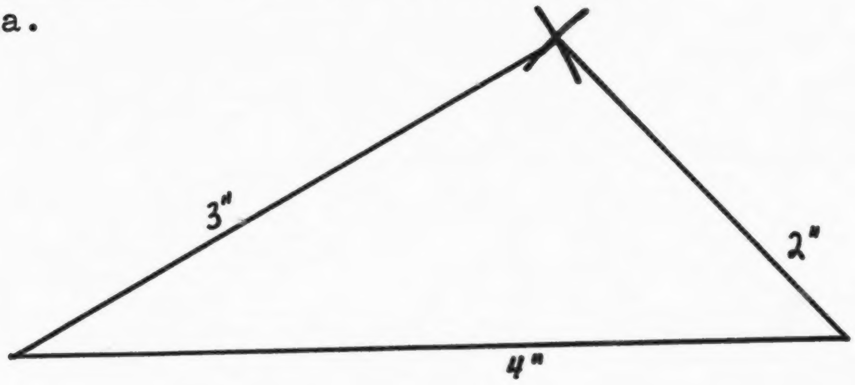
15. 3240°

16. $S = (N - 2)\ 180^\circ$

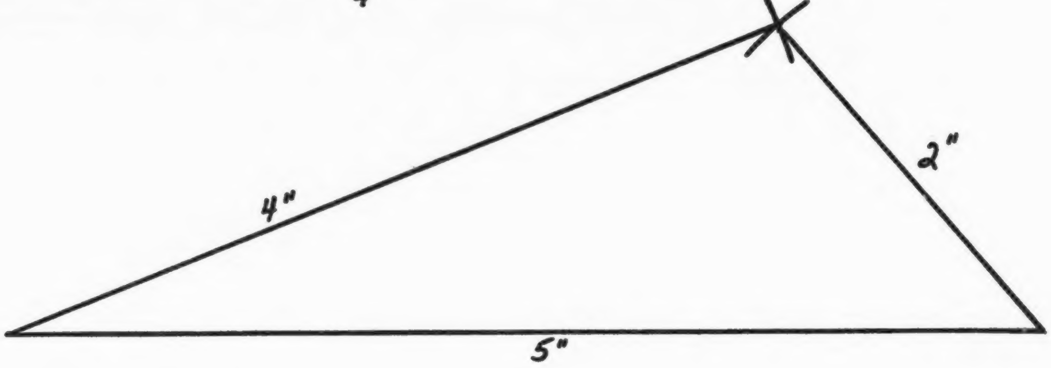
17. 360°

Pages 59-61 Class Discussion-11

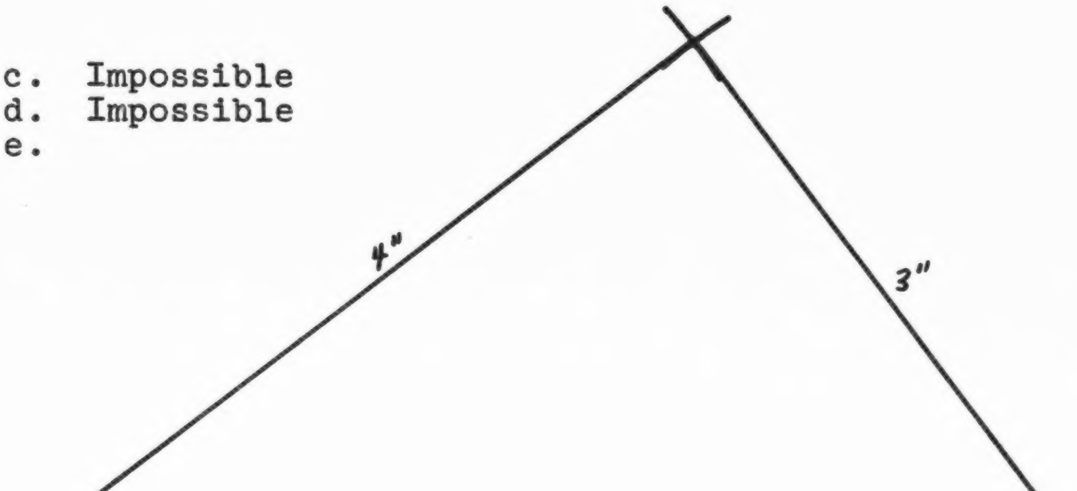
1a.



b.



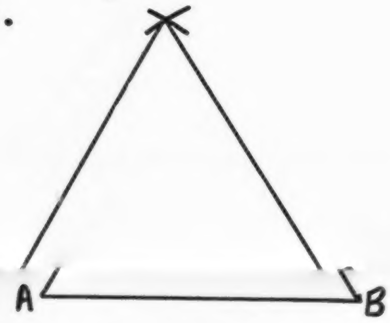
c. Impossible
d. Impossible
e.



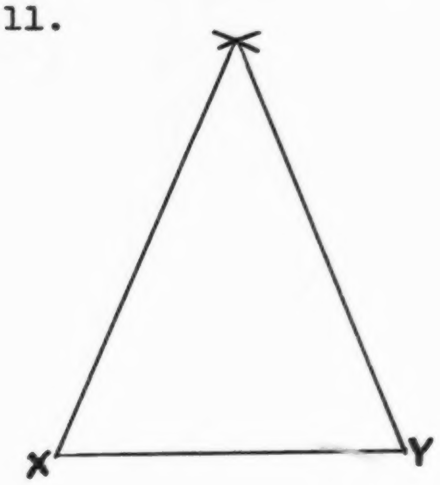
2. No * The two shorter sides are not long enough to meet.

Class Discussion-11 (continued)

3. No * The two shorter sides together are the same length as the longest side. Thus they cannot meet in a point that is outside the segment having the greatest length.
4. Yes * A right triangle
5a. Yes
b. Yes
c. Yes
d. No
e. Yes
f. Yes
g. No
h. No
i. No
j. Yes
6. Answers vary. (The sum of the lengths of the two shorter sides must be less than the length of the longest side.)
7. a, b, f, j
8. a, j
9. a, b
10.

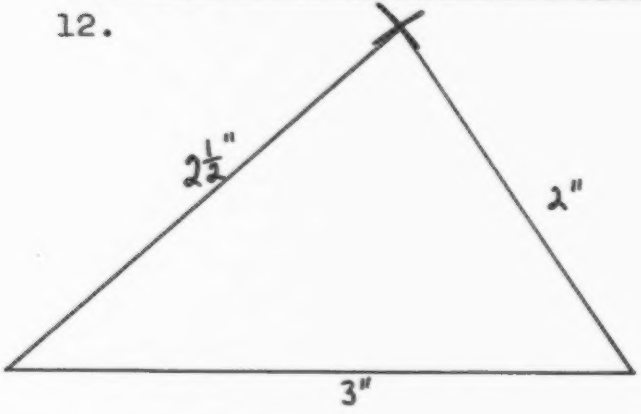


11.



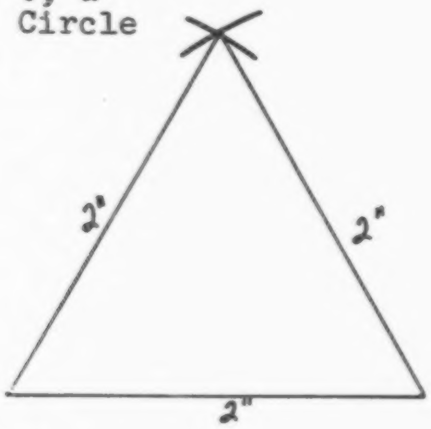
Class Discussion-11 (continued)

12.

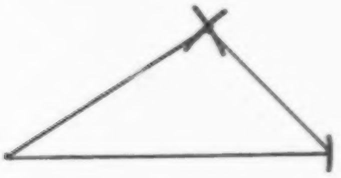


Page 61 Exercises-11

1. CD, EF, KL, MN
2a. p
b. q, r, s, v
c. t, u
d. Circle
3a.

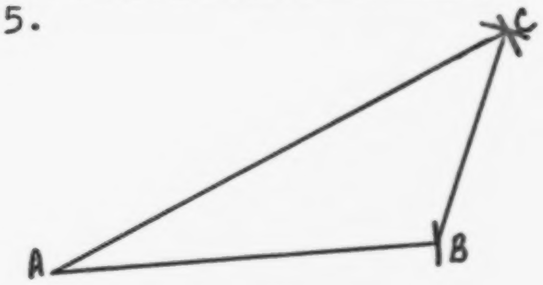


b. Equilateral
4.



Exercises-11 (continued)

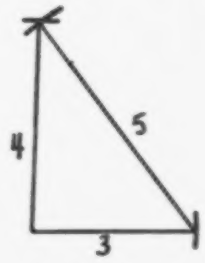
5.

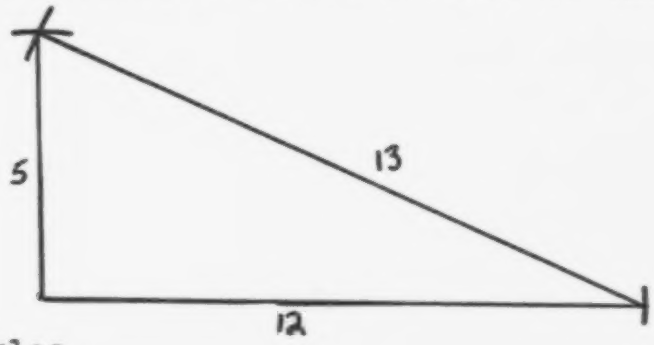


6a. Not possible

b. The sum of the measures of any two sides of a triangle must be greater than the measure of the third side.

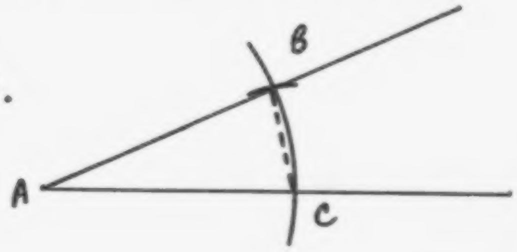
7.



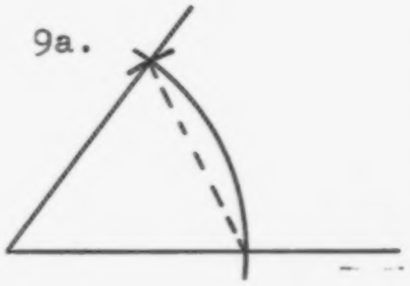


They are right triangles.

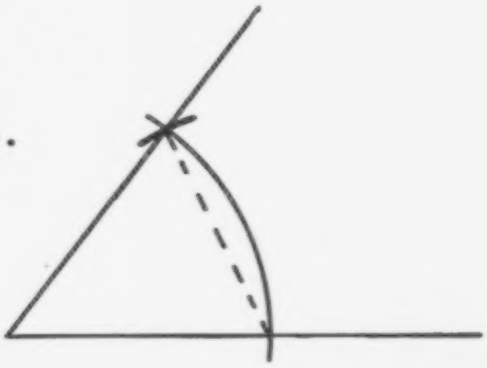
8.



9a.

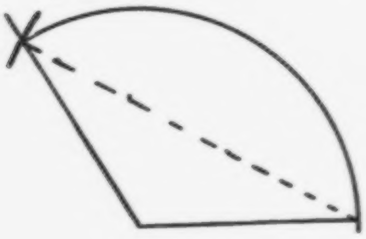


b.

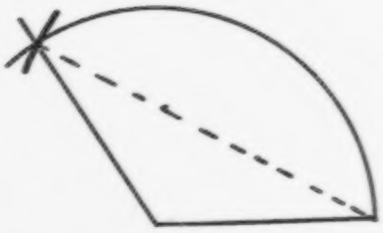


Exercises-11 (continued)

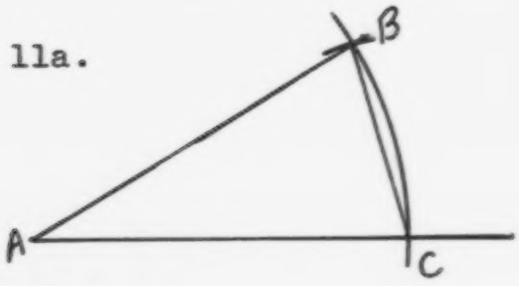
10a.



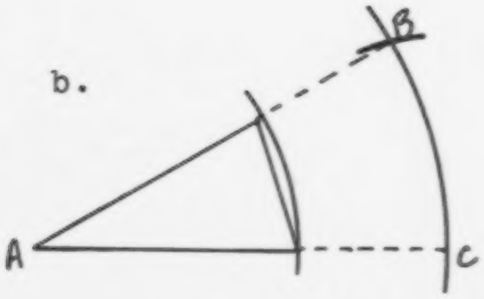
b.



11a.

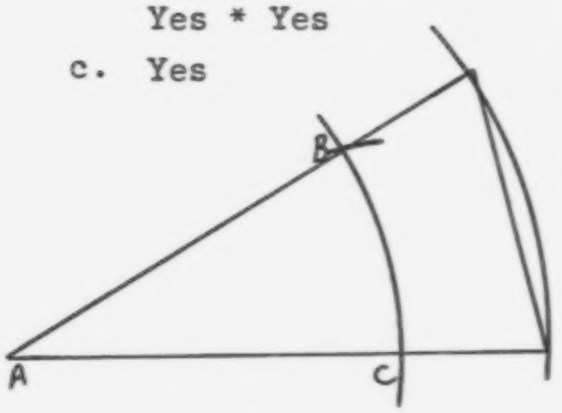


b.



Yes * Yes

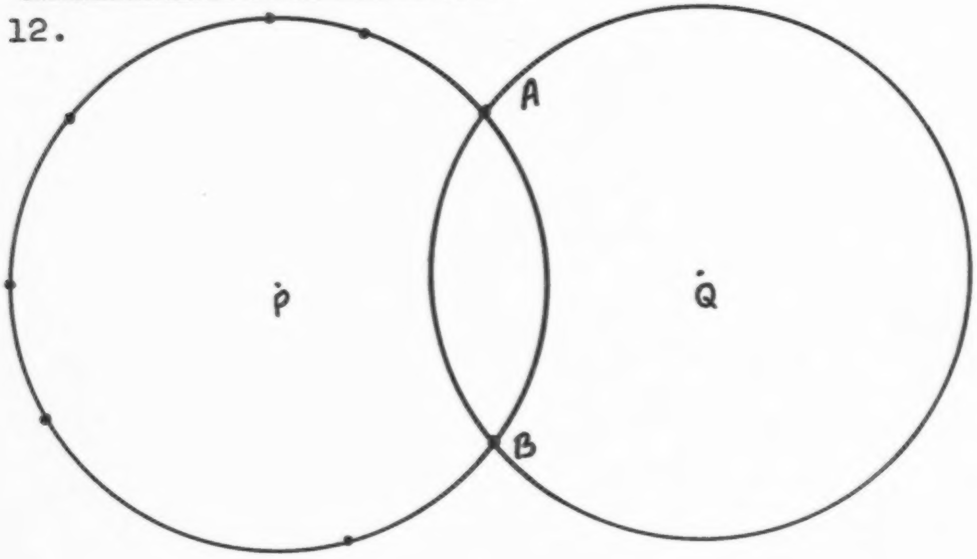
c. Yes



d. An infinite number

Exercises-11 (continued)

12.

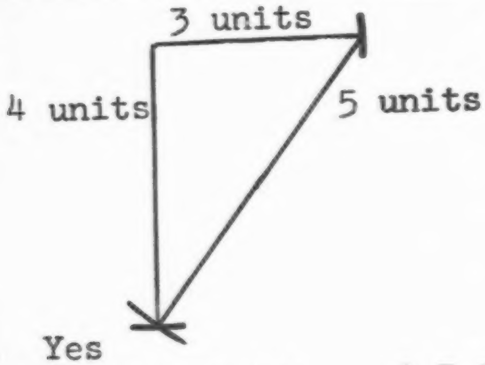


d. Circle
g. Points A and B are each 5 units from both P and Q.

13. Drawings vary.

Pages 65-66 Class Discussion-12a

1a.

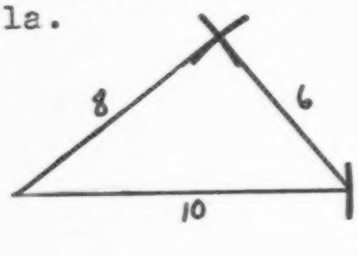

3 units
4 units
5 units

b. Yes
c. Sides can be: 2-5-5, 3-4-5, or 4-4-4.
d. No

2. No

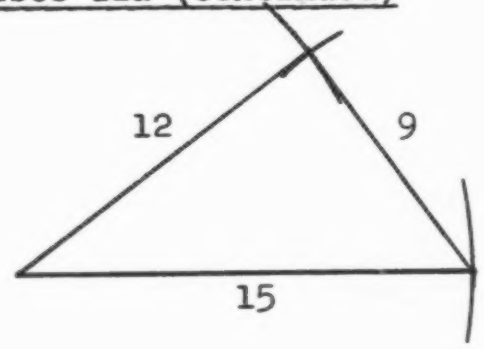
Pages 66-68 Exercises-12a

1a.

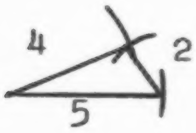


Exercises-12a (continued)

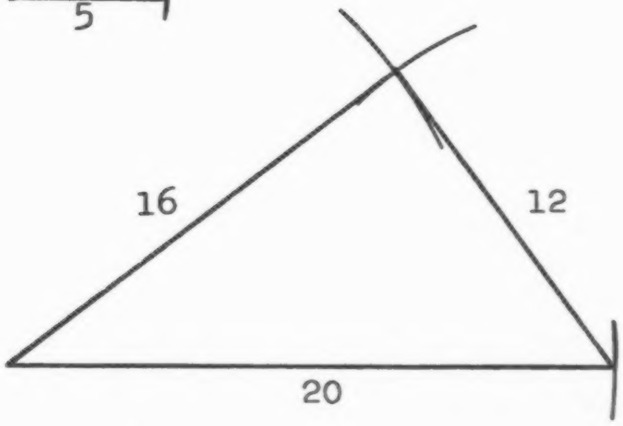
1b.



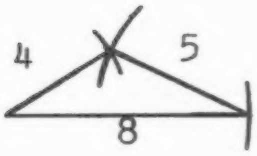
c.



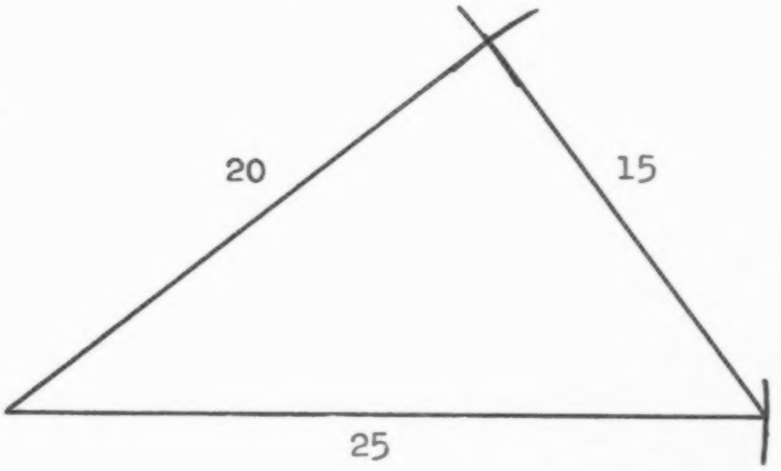
d.



e.

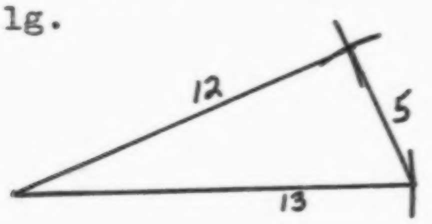


f.

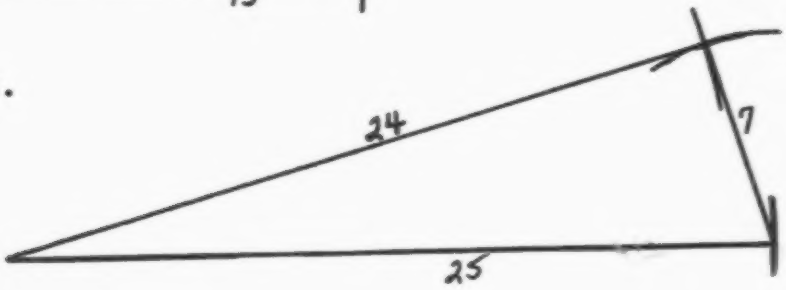


Exercises-12a (continued)

1g.



h.



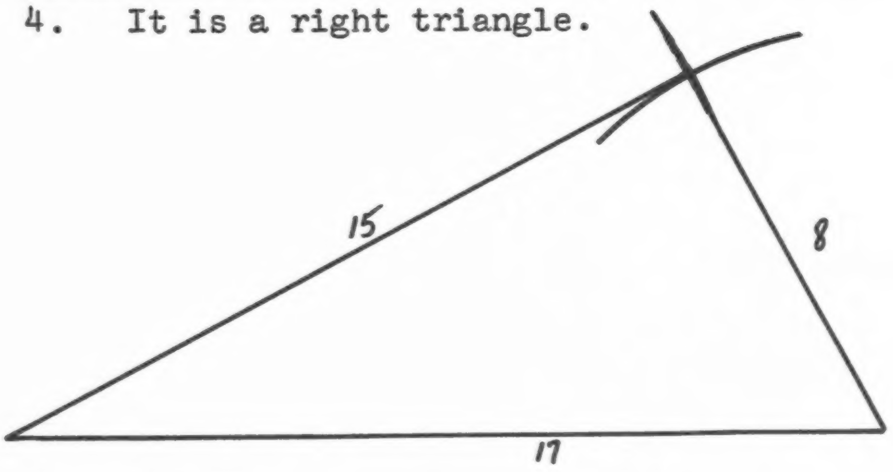
2. 3-4-5, 6-8-10, 9-12-15, 12-16-20, 15-20-25, 5-12-13, 7-24-25

3.

Lengths of the sides of a Right Triangle		
Shortest Side	Second Side	Longest Side
6	8	10
15	20	25
18	24	30
21	28	35
30	40	50
300	400	500

Exercises-12a (continued)

4. It is a right triangle.



Pages 68-69 Class Discussion-12b

1a. $9 + 16 = 25$.
 b. $36 + 64 = 100$.
 c. Yes
 d. Yes
 e. The square of the longest side of a right triangle is equal to the sum of the squares of the two shorter sides. * The relationship does hold for entries in exercise 3.
2. Yes

Pages 70-72 Exercises-12b

1. Yes * No * No * Yes
2.

c	4	5	17	20	29	$\frac{5}{2}$	40
c^2	16	25	289	400	841	$\frac{25}{4}$	1600

3a. $c^2 = 289$; $c = 17$.
 b. $c^2 = 1600$; $c = 40$.
 c. $c^2 = 841$; $c = 29$.
 d. $c^2 = \frac{25}{4}$; $c = \frac{5}{2}$.

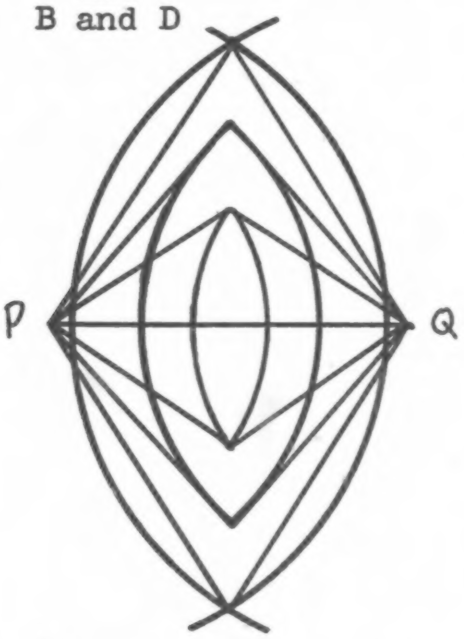
4a. 3 and 4
 b. 7 and 8
 c. 9 and 10
 d. 13 and 14

Exercises-12b (continued)

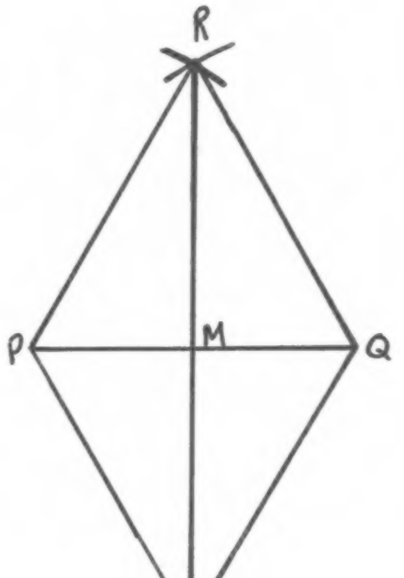
5a. 12
b. 11
c. 14
d. 21
6. 15 in.
7. 150 ft.
8. 58 miles
9. 5 ft. (approximately)
10. 13

Pages 73-76 Class Discussion-13

1. A and C
2. B and D
3.

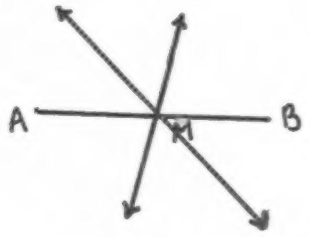


a. No
b. The radius must be longer than half the length of PQ.



Class Discussion-13 (continued)

4. PM and MQ are equal in length.
5. RM and MS are equal in length.
6. Line PQ * Yes
7. Equal
8. They are perpendicular to each other.
9. (Several constructions) * They are equal right angles in each case.
10.

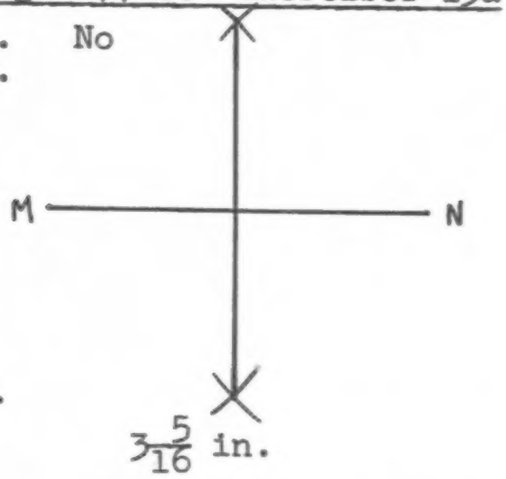


An infinite number

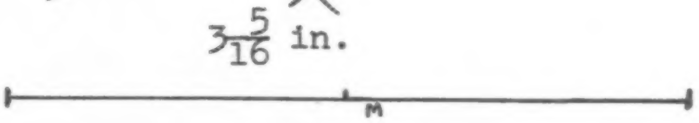
11. An infinite number
12. One
13. They are equal. (about 5
14. They are equal. (about 58°)
15. They are equal. (about 32°)
16. They are equal. (about 32°)
17. Yes

Pages 77-80 Exercises-13a

1. No
2.

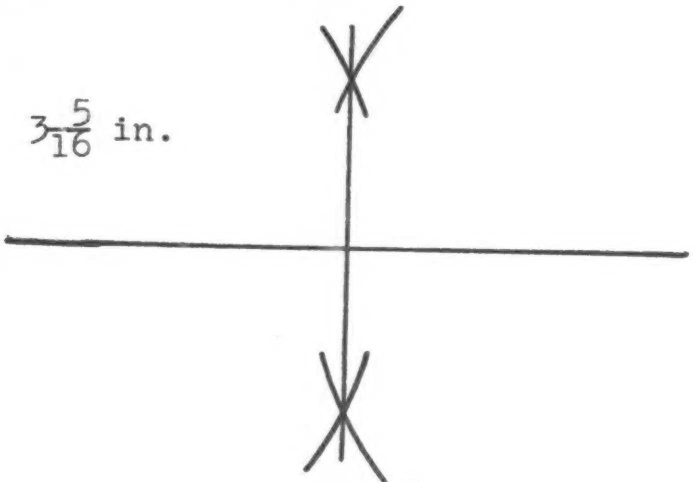


3.

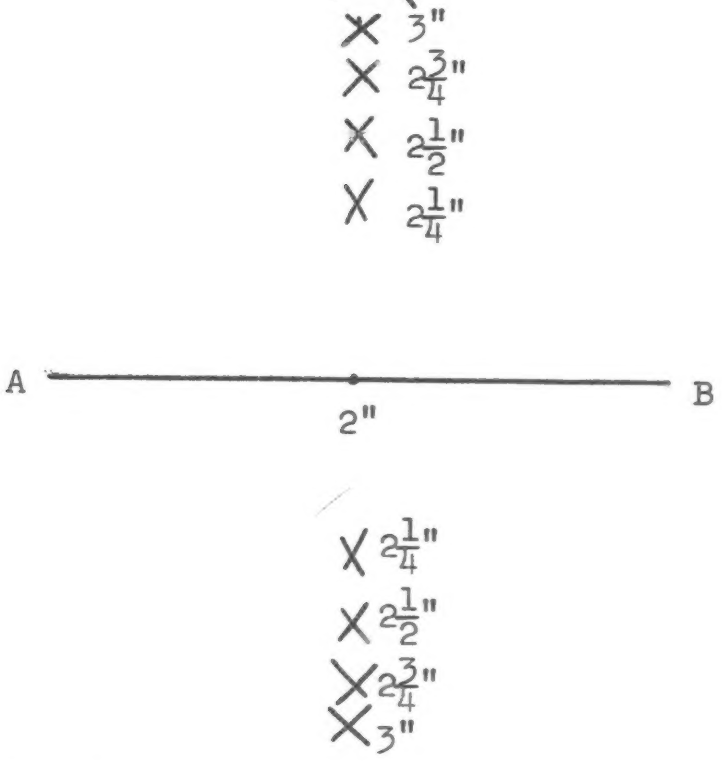


Exercises-13a (continued)

4.

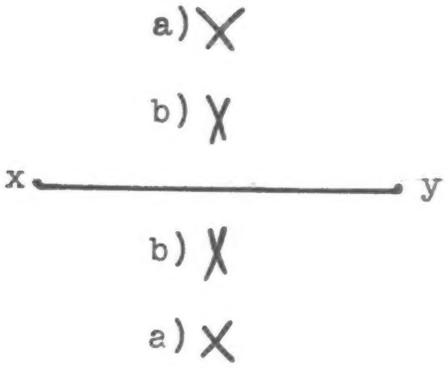

$3\frac{5}{16}$ in.

5a.


3"
$2\frac{3}{4}$"
$2\frac{1}{2}$"
$2\frac{1}{4}$"
A
B
2"
$2\frac{1}{4}$"
$2\frac{1}{2}$"
$2\frac{3}{4}$"
3"

b. A and B are more than 3 inches apart.

6a-b.

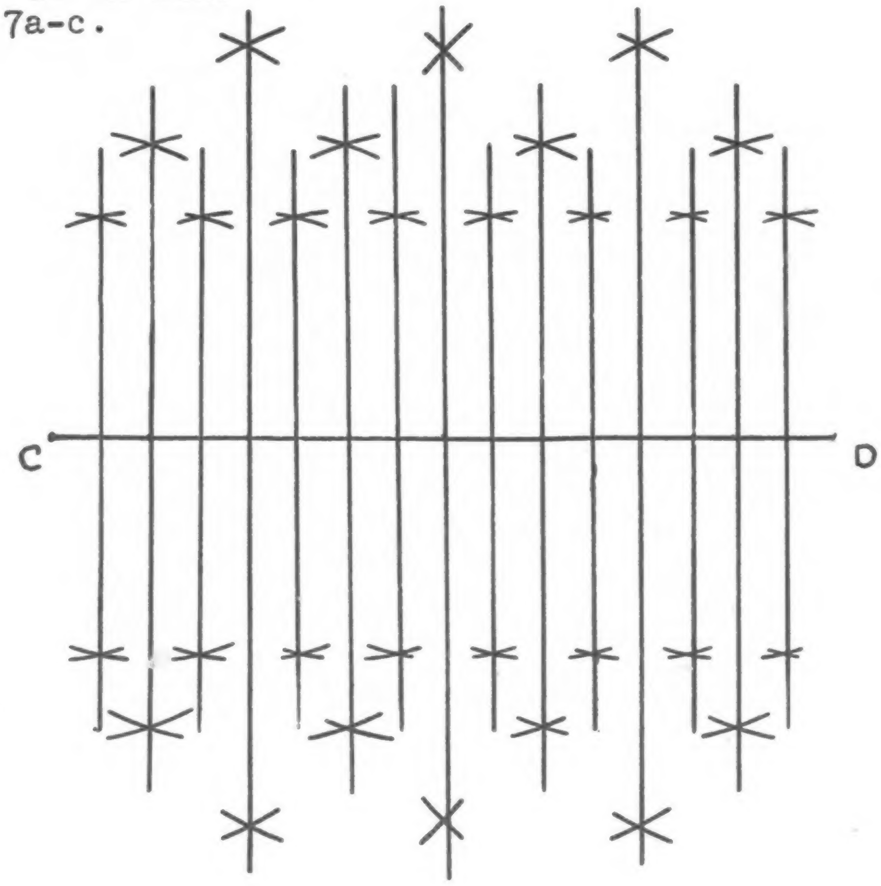


Exercises-13a (continued)

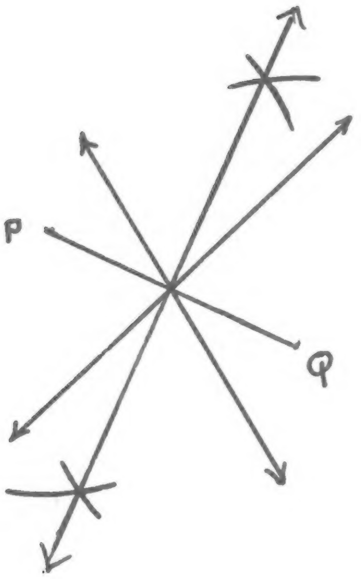
6c. An infinite number

d. A line

7a-c.

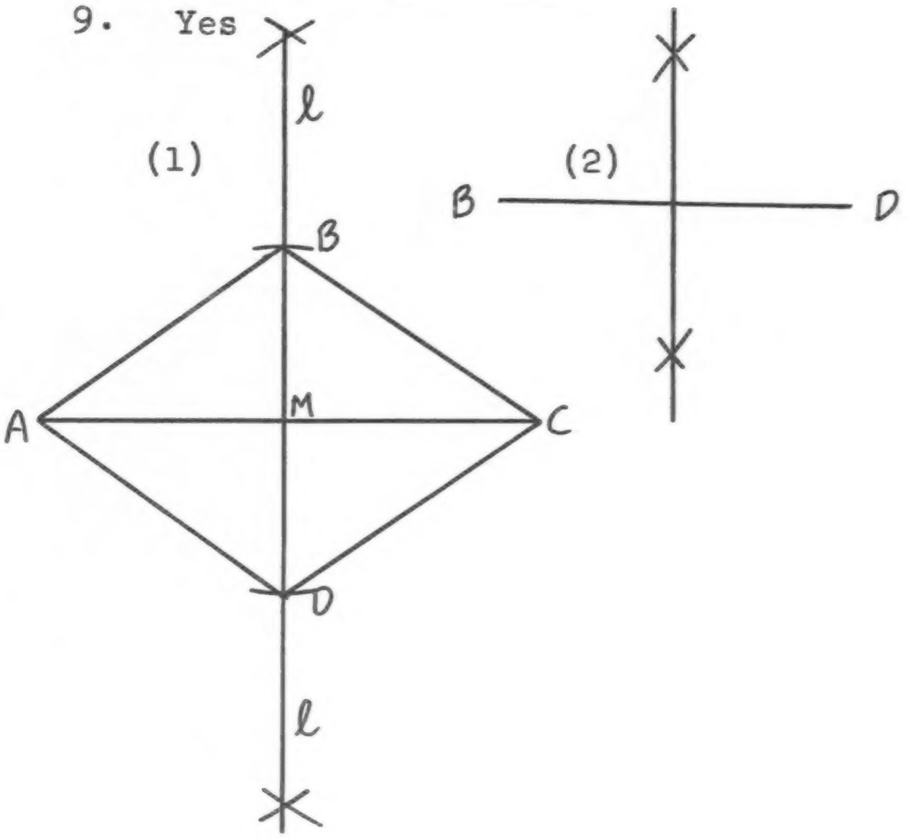


8.



Exercises-13a (continued)

9. Yes



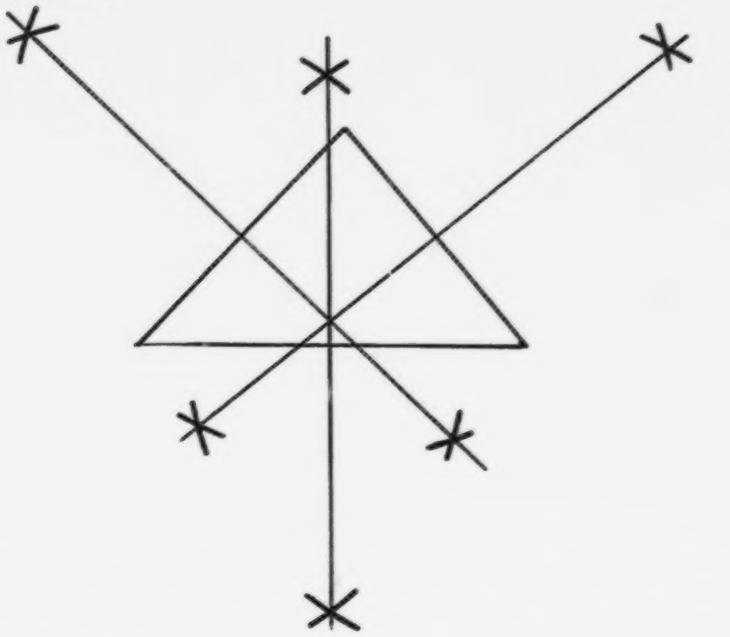
First, construct line ℓ which is the perpendicular bisector of segment AC (see figure (1)).

Second, copy segment BD and locate its midpoint (see figure (2)).

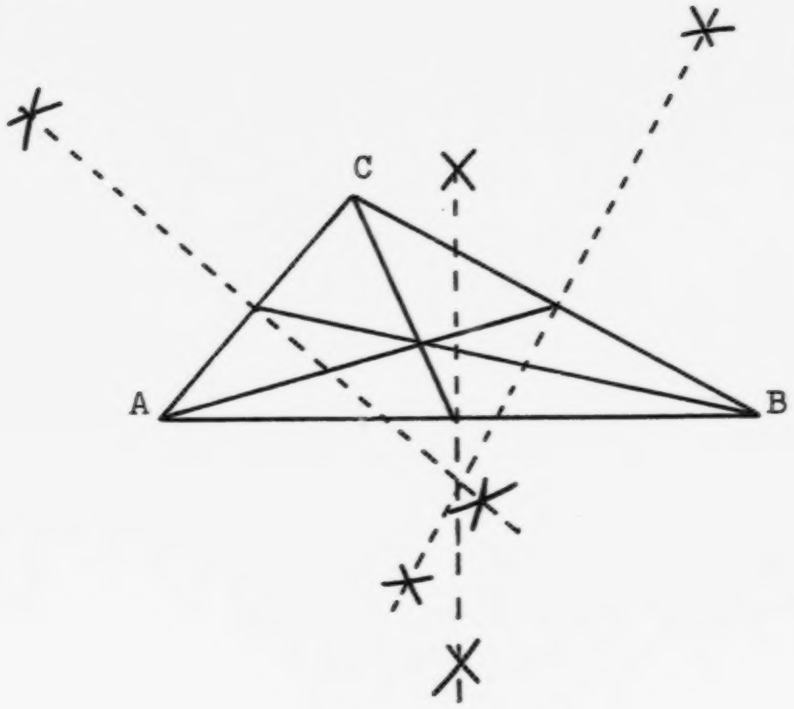
Third, on the two sides of M on line ℓ construct segments MB and MD having lengths equal to half of the length of segment BD.

Exercises-13a (continued)

10a-b.

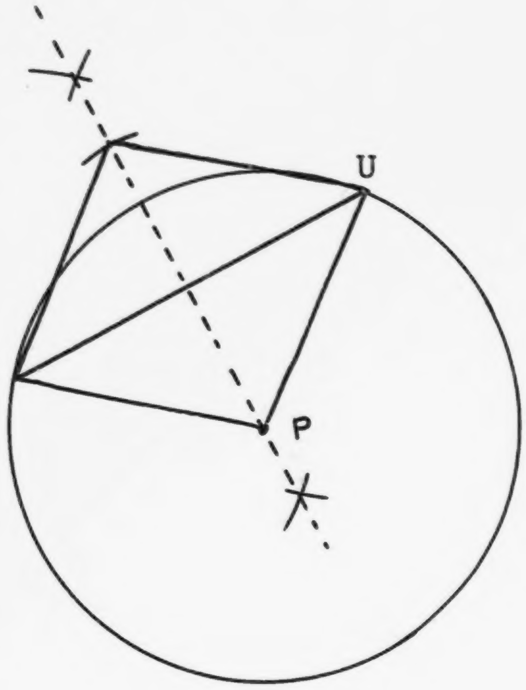


11a-c.



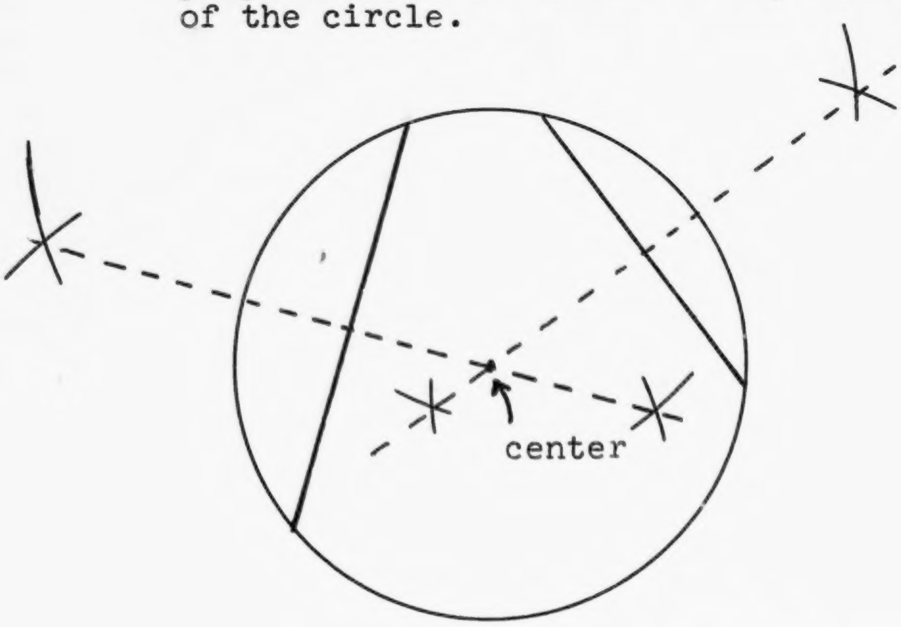
Exercises-13a (continued)

12.



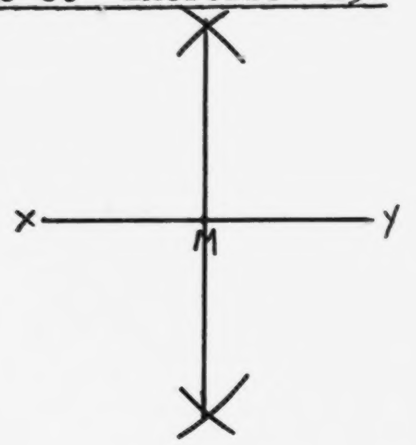
Yes

13. Draw two chords of the circle. Construct their perpendicular bisectors. They meet at the center of the circle.

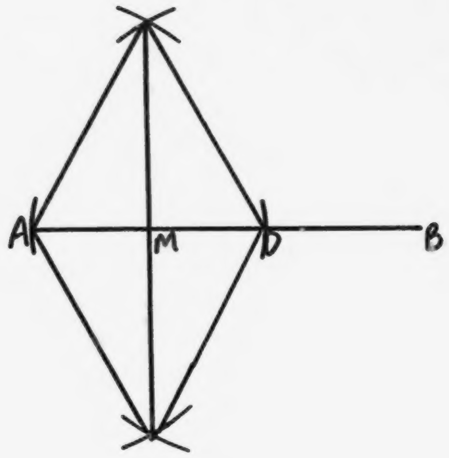


Page 80 Exercises-13b

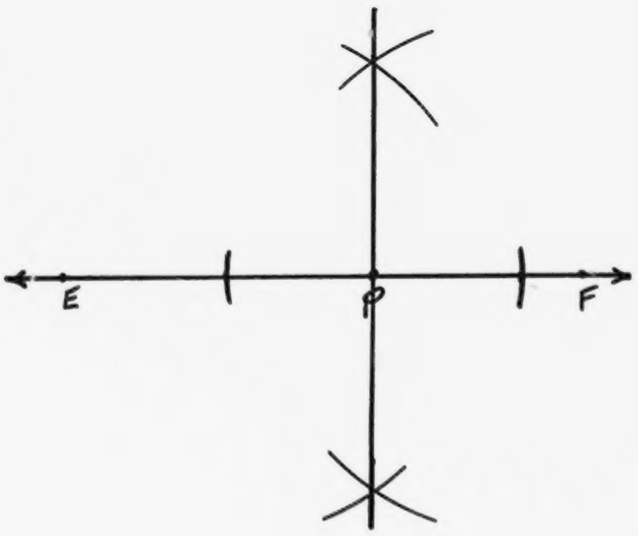
1.



2.

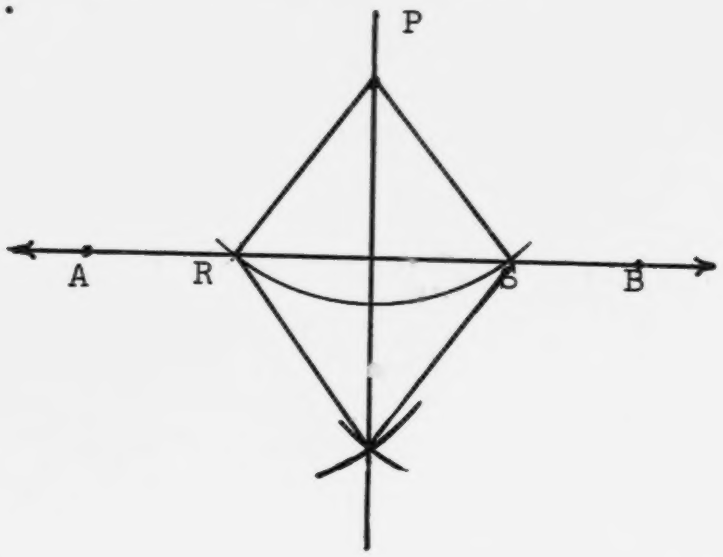


3.

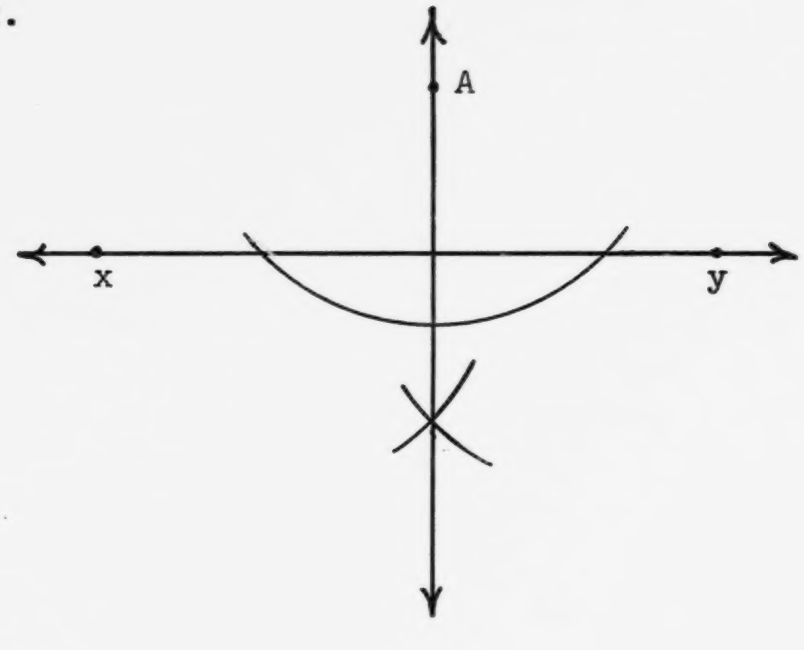


Pages 81-84 Exercises-13c

1.

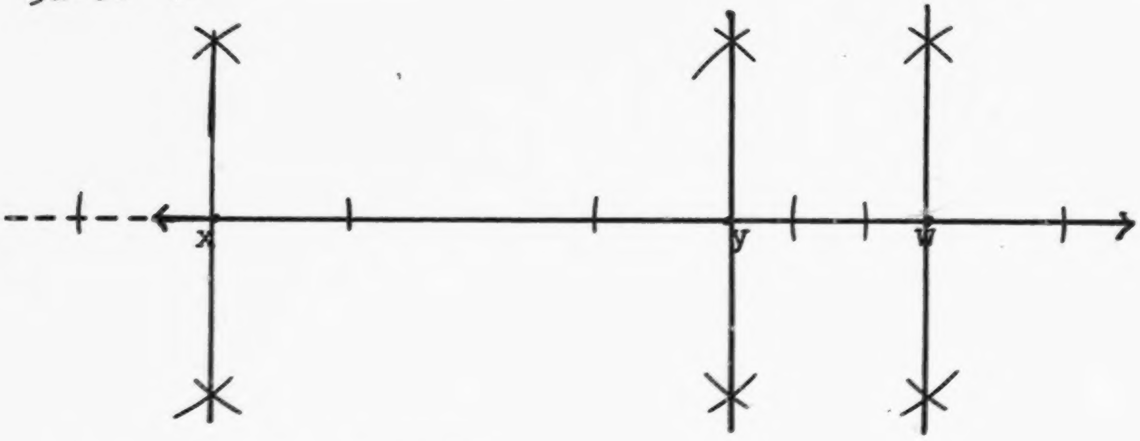


2.



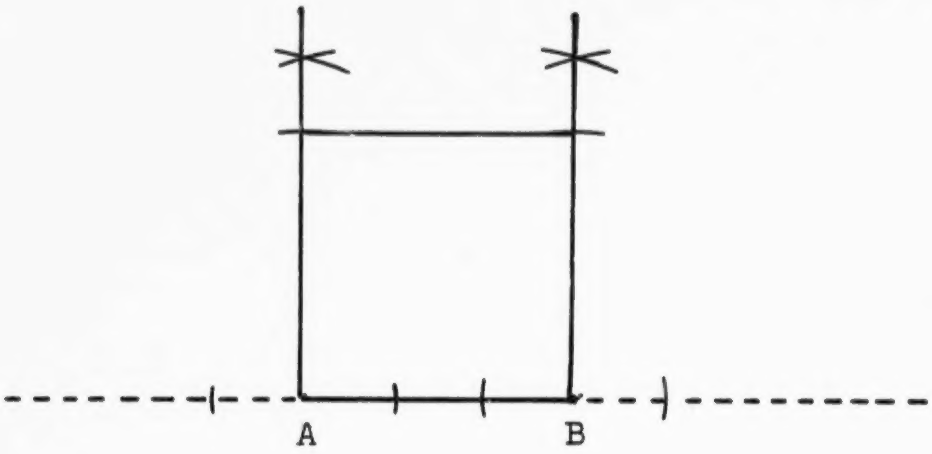
Exercises-13c (continued)

3a-c.

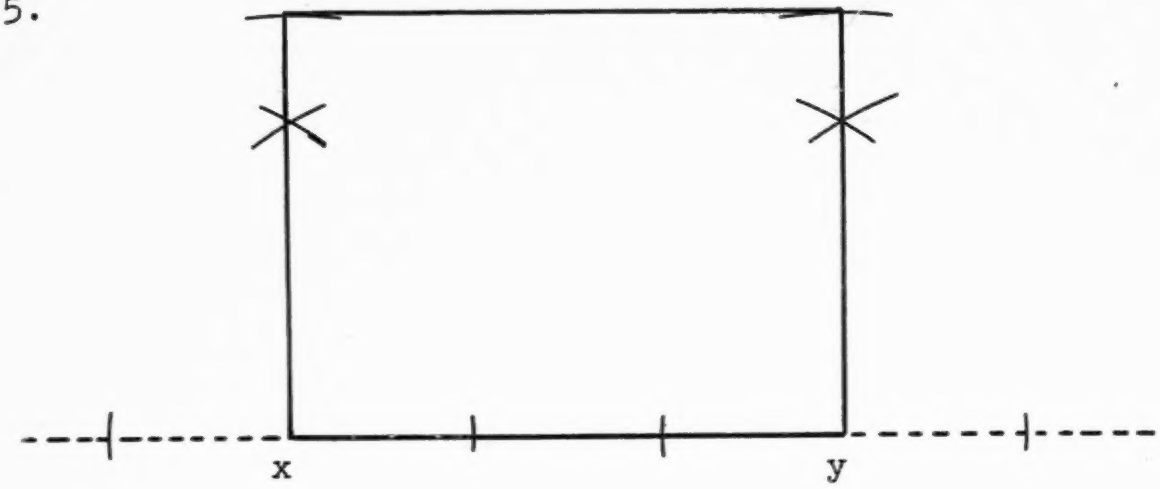


d. They are parallel.

4.

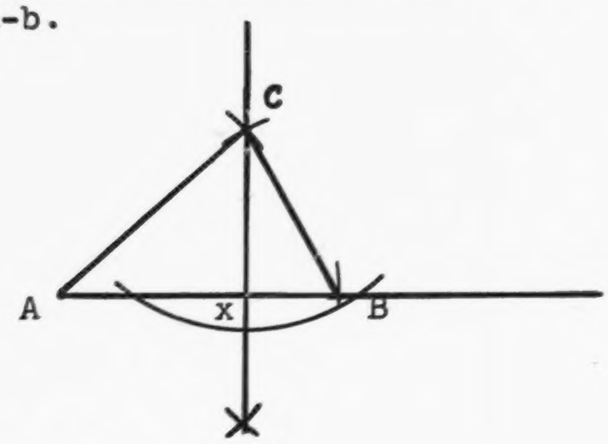


5.



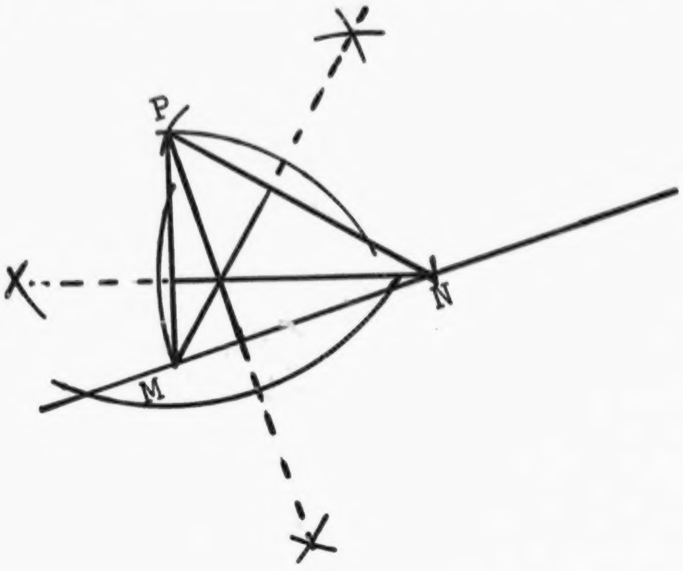
Exercises-13c (continued)

6a-b.



c. Yes * One altitude could be drawn from B to side AC, and one from A to side BC.

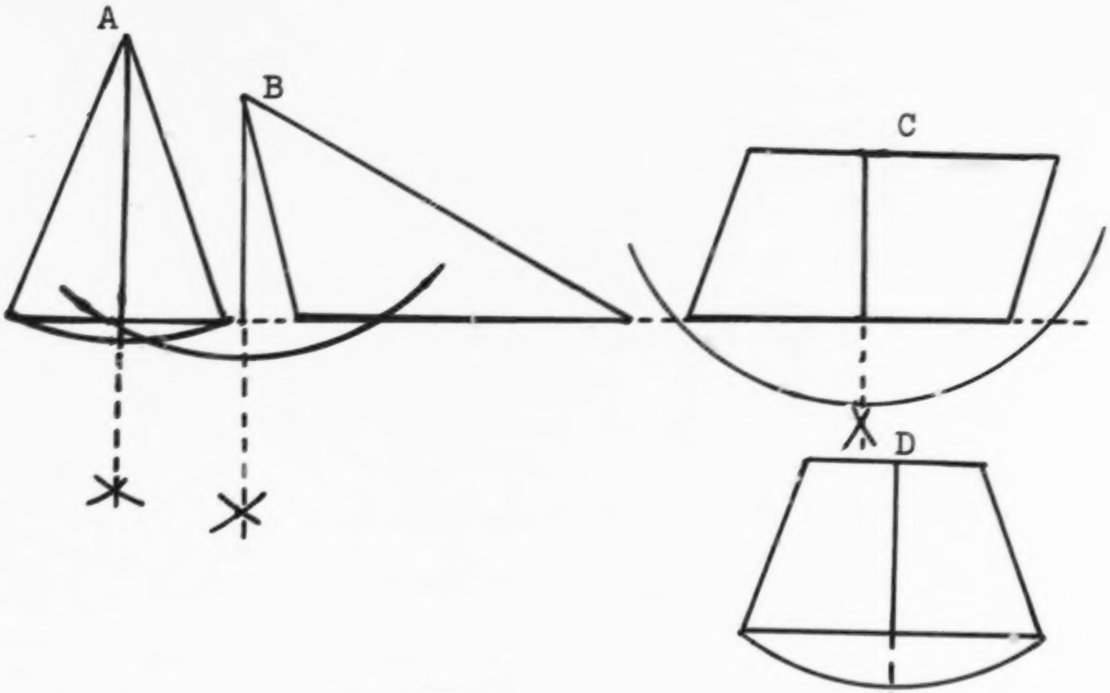
7.



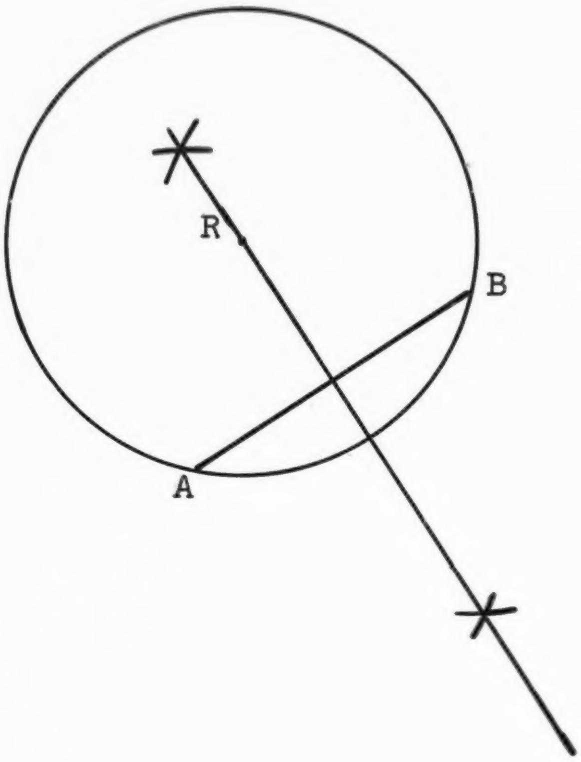
8a. A

Exercises-13c (continued)

8b.



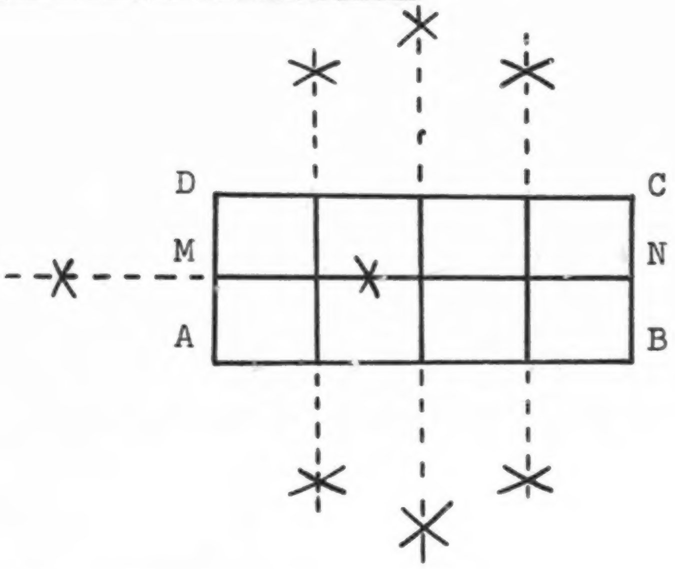
9a-c.



d. Yes

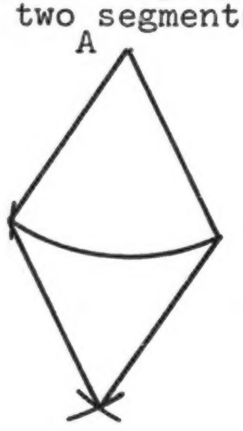
Exercises-13c (continued)

10.

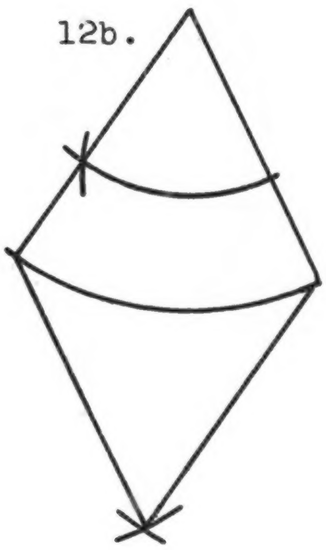


(1) Construct the perpendicular bisector of segment AD.
(2) Construct the perpendicular bisector of the segment MN located in step (1).
(3) Construct the perpendicular bisector of each of the two segments obtained in step (2).

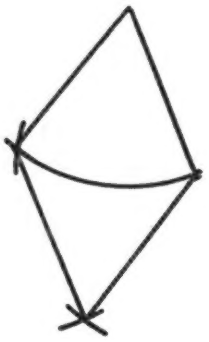
11.



12b.

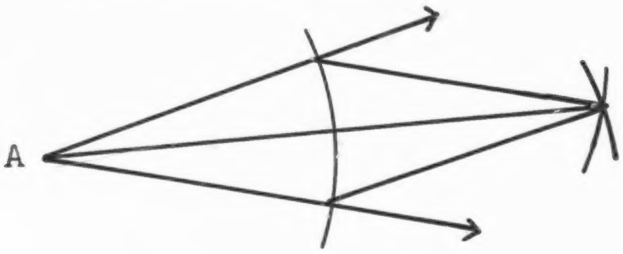


12a.

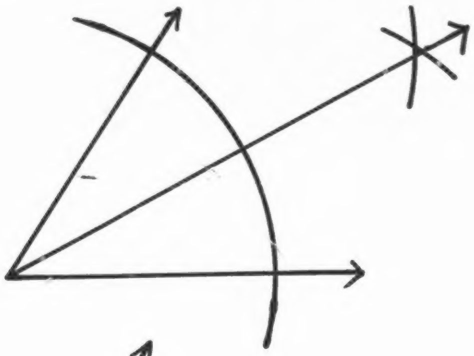


Pages 84-87 Exercises-13d

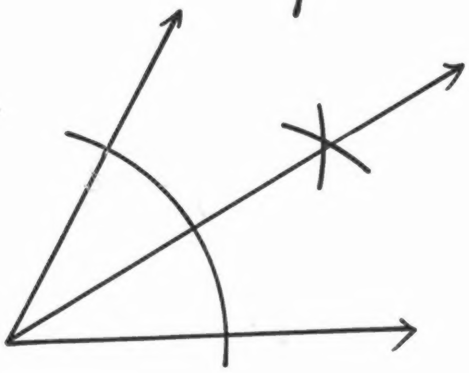
1.



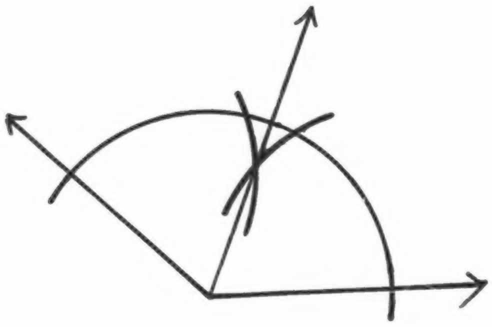
2.



3.

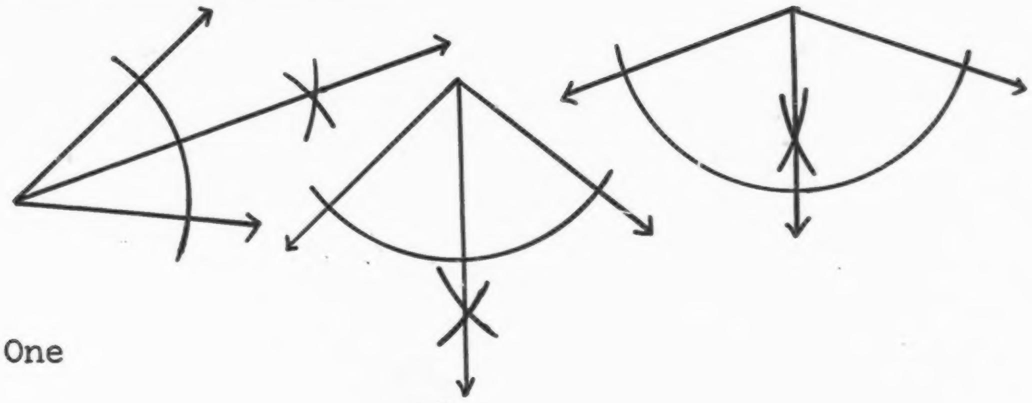


4.



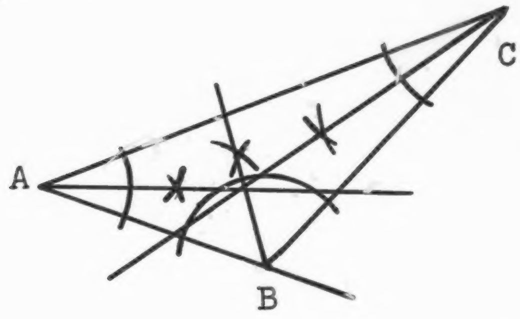
Exercises-13d (continued)

5a.

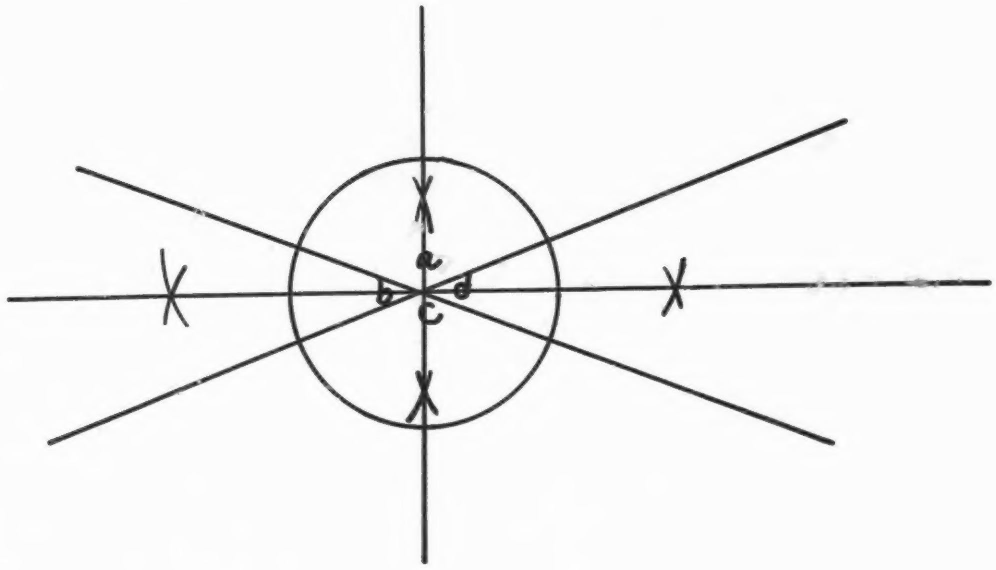


b. One

6.



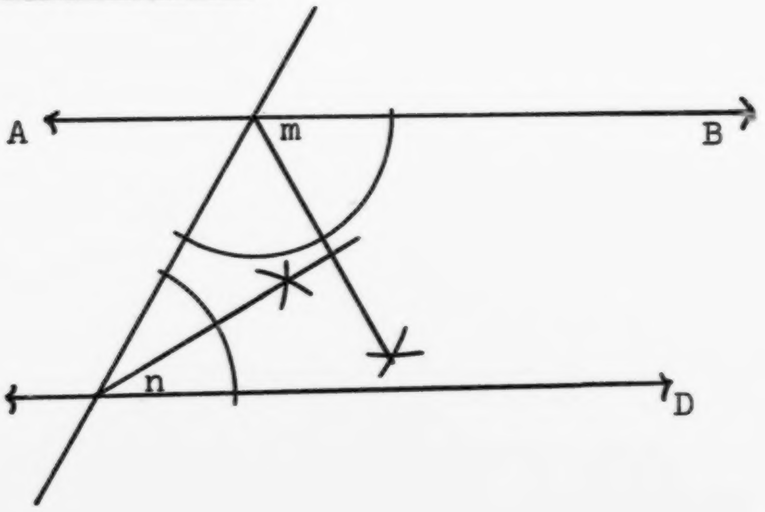
7a-b.



c. The bisectors of angles a and d form a right angle.
d. The bisectors of angles b and d are opposite rays.
e. Yes

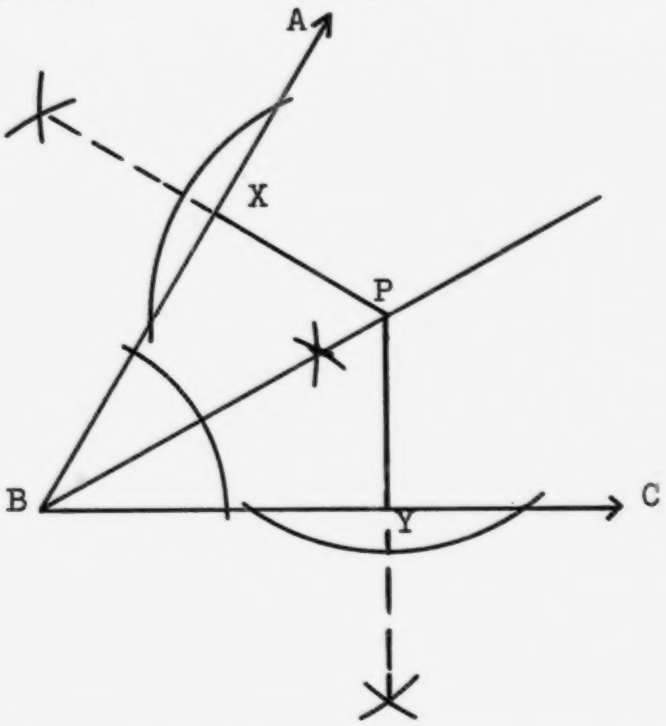
Exercises-13d (continued)

8a-c.



d. They appear to be perpendicular.

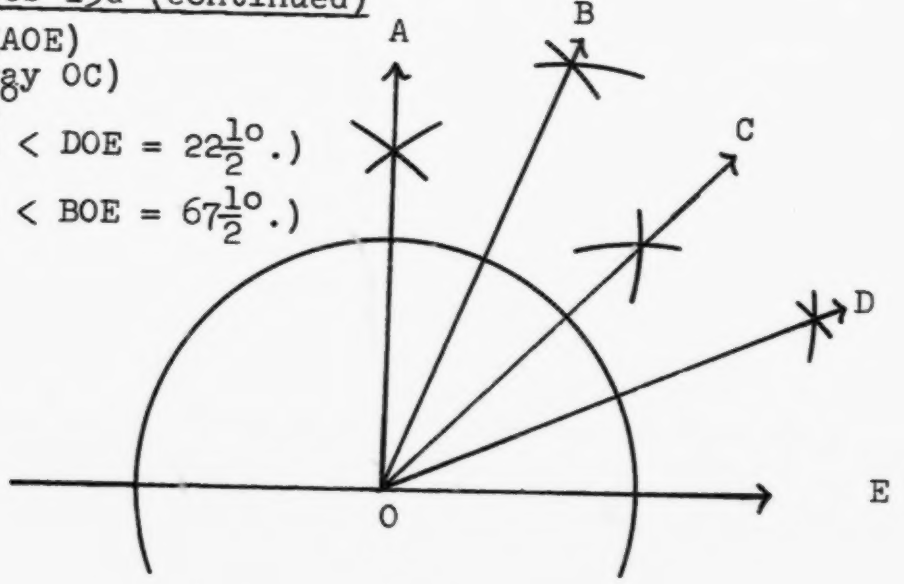
9.



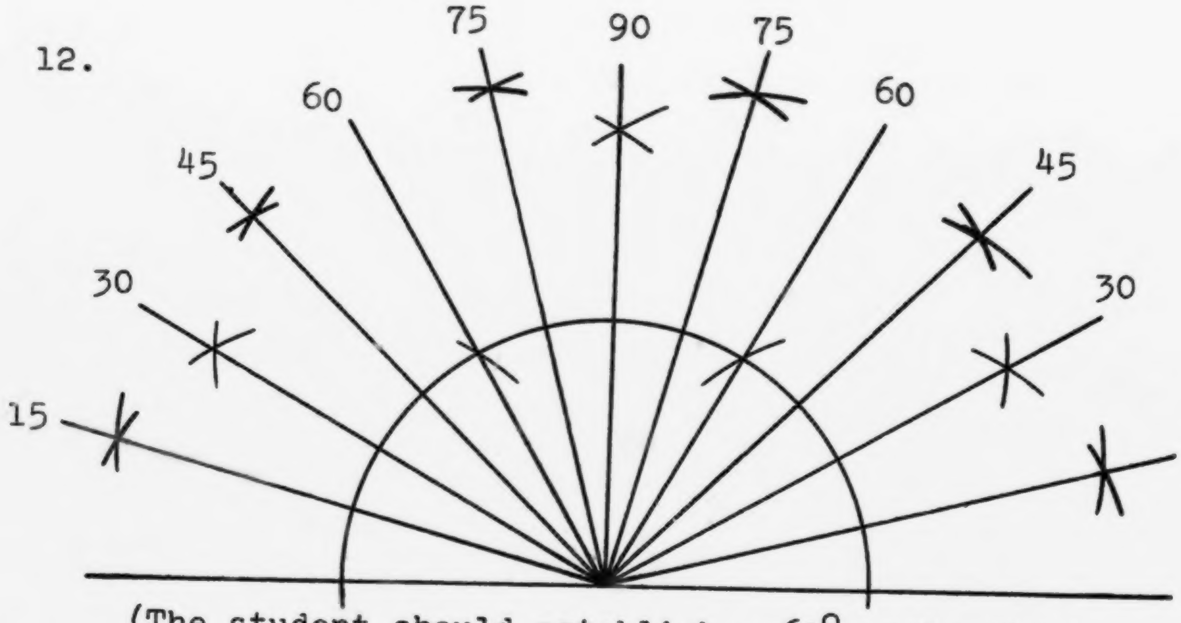
10. Locate the instrument so that the wire passes through the vertex of the angle and each end of the strip of wood touches a side of the angle. Then the wire indicates the angle bisector.

Exercises-13d (continued)

11a. (<AOE)
b. (ray OC)
c. 45°
d. ($m < DOE = 22\frac{1}{2}^{\circ}$.)
e. ($m < BOE = 67\frac{1}{2}^{\circ}$.)



12.


75
90
75
60
60
45
45
30
30
15

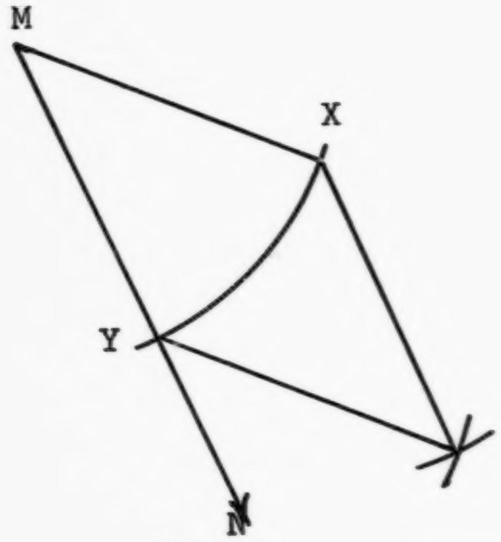
(The student should establish a 60° angle first by constructing an equilateral triangle.)

Pages 87-91 Exercises-13e

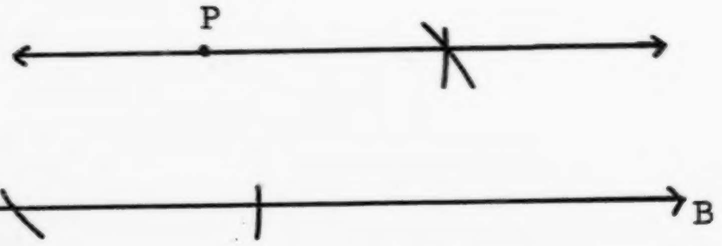
1. AB and DC, or AD and BC

Exercises-13e (continued)

2.

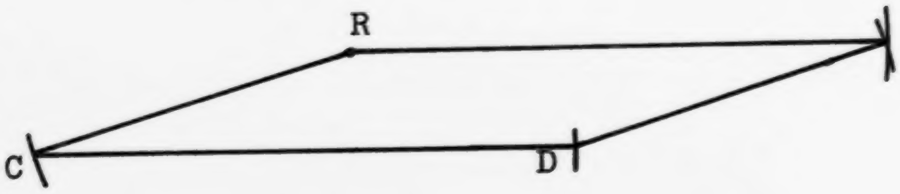


3.

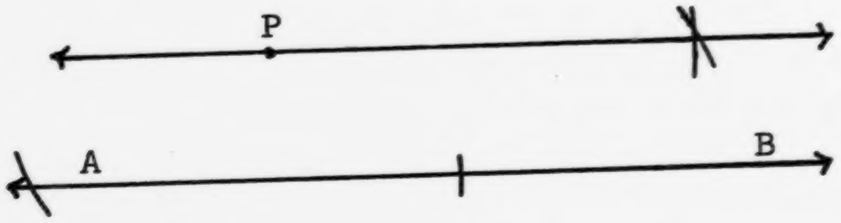


4. Not all four sides are congruent.

5.



6.

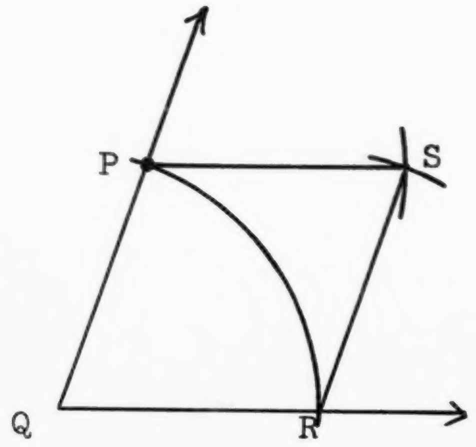


7a. (1) and (2)
  b. Yes
  c. Yes
  d. Yes

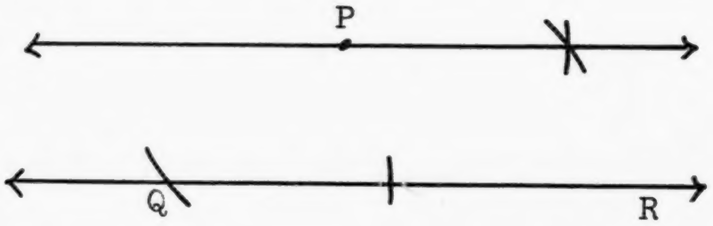
Exercises-13e (continued)

8. Quadrilaterals ABGH, ACFH, ADEH, BCFG, BDEG, CDEF

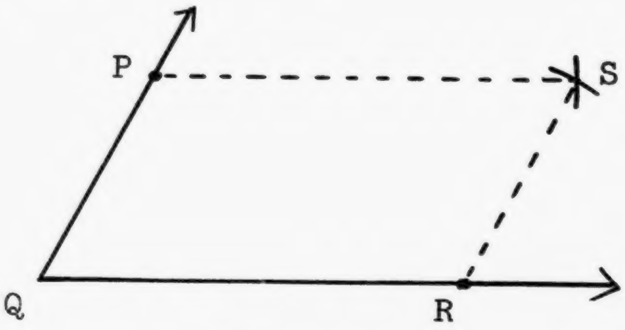
9.



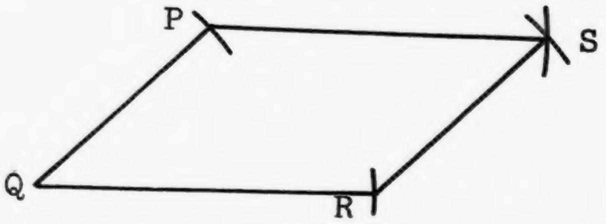
10.



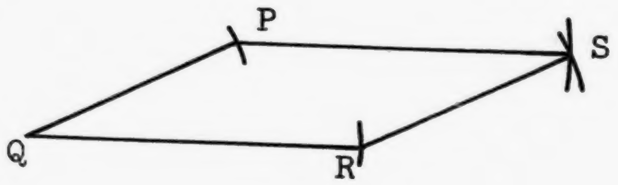
11.



12a.

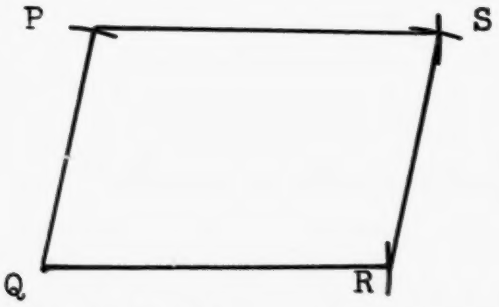


b.



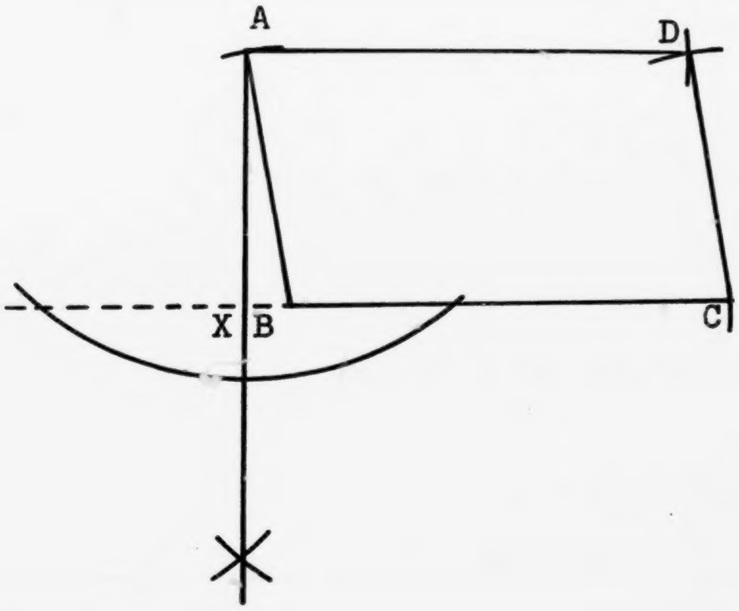
Exercises-13e (continued)

12c.



d. An infinite number

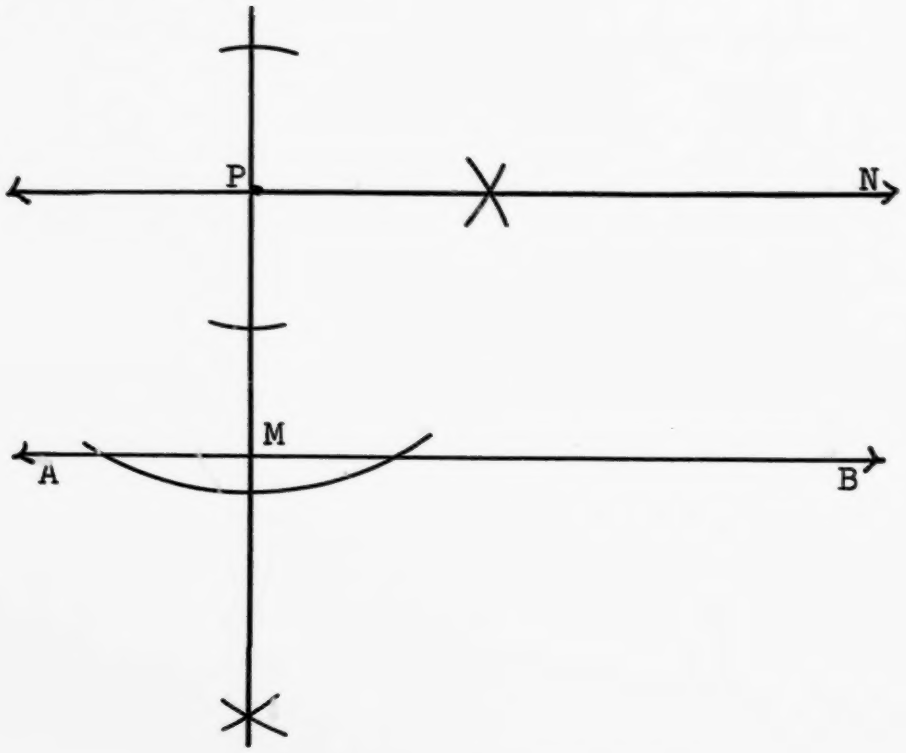
13a-b.



c. No * As angle B becomes more obtuse, the altitude would become shorter.

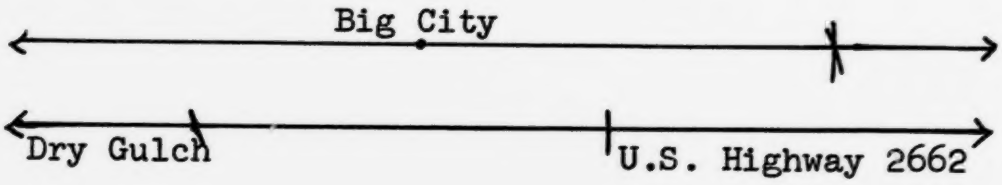
Exercises-13e (continued)

14a-b.



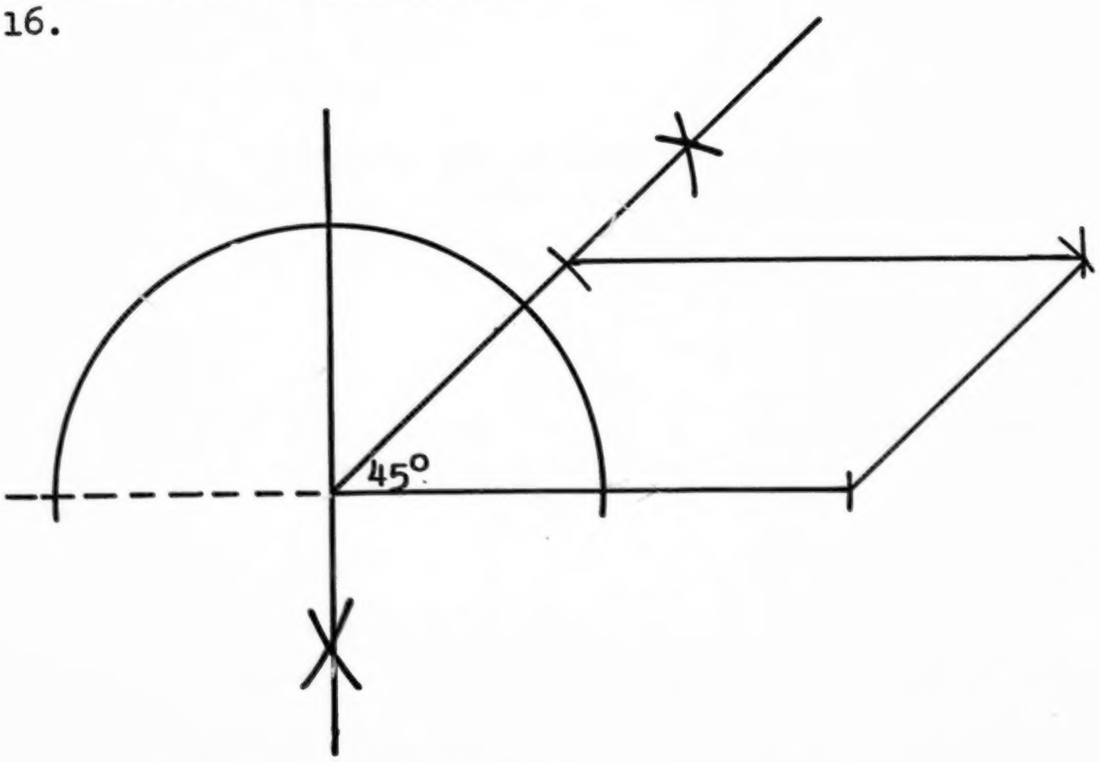
c. Line PN is parallel to line AB.

15.


Big City
Dry Gulch
U.S. Highway 2662

Exercises-13e (continued)

16.



Page 94 Review Exercises

1a. Segments BD and AC
b. <ABC, or <BCD, or <CDA, or <DAB
c. Sides AD and BC, or sides AB and DC
d. <AEB, <BEC, <CED, <DEA (any two of these)
e. Segments BD and AC
f. E
g. Segments AB, BC, CD, and DA
h. <BCA, <DCA, <BCD
i. <BAE (Answers vary.)
j. Triangle ABC, ADC, BCD, or BAD
k. Triangle BAD or BCD
l. Segment DE

2. (1) a, c, d, e, f, g
(2) a, c, d, e, f, g
(3) f
(4) a
(5) a, c, d, g
(6) a, g

Review Exercises (continued)

2. (7) a, c
   (8) b, h
   (9) b, h
   (10) h

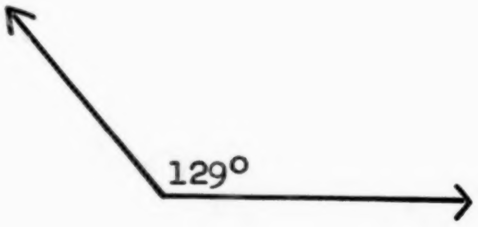
3a. The faces bounded by triangles DEF and ABC
b. Segments AC and BE (Answers vary.)
c. A, B, C, D, E, and F
d. Segments AB, BC, CA, DE, EF, FD, AD, CF, BE
e. Faces ABED, ACFD, BEFC

4. $m < ABC = 48^\circ$

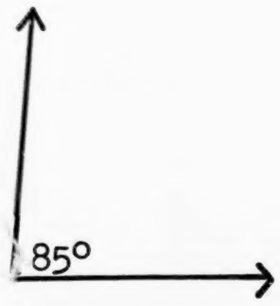
$m < CBD = 90^\circ$

$m < ABD = 138^\circ$

5a.

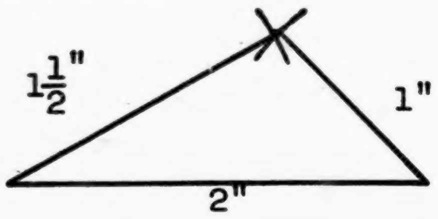


b.



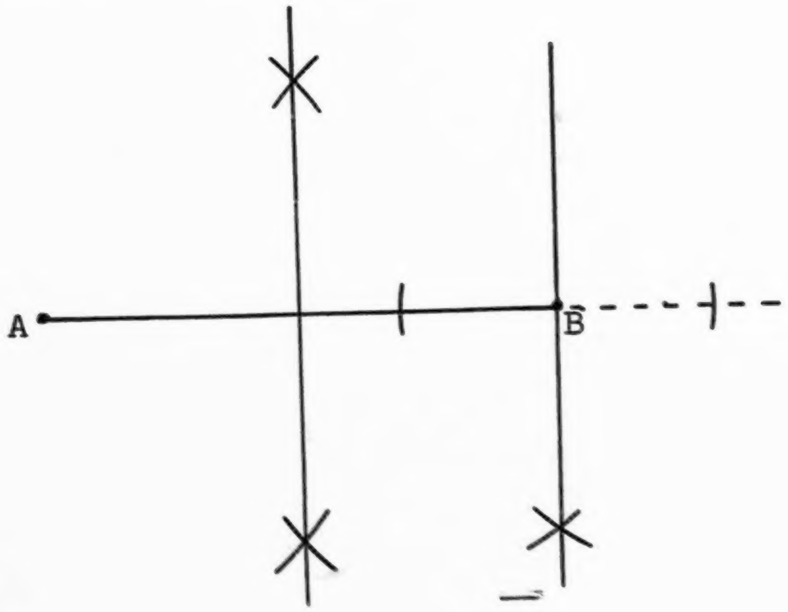
6a. 46°
b. 36°

7.



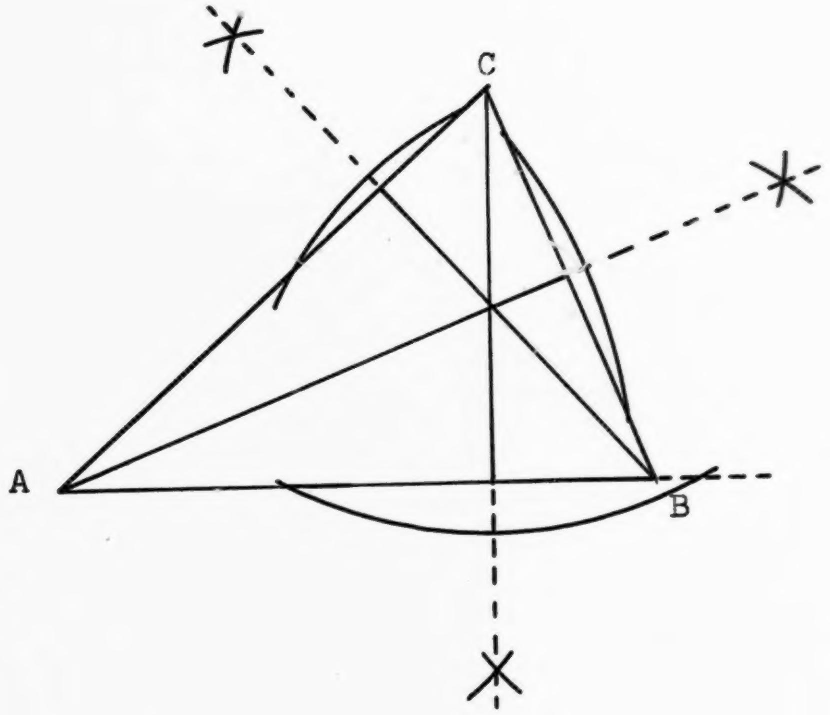
Review Exercises (continued)

8a-c.



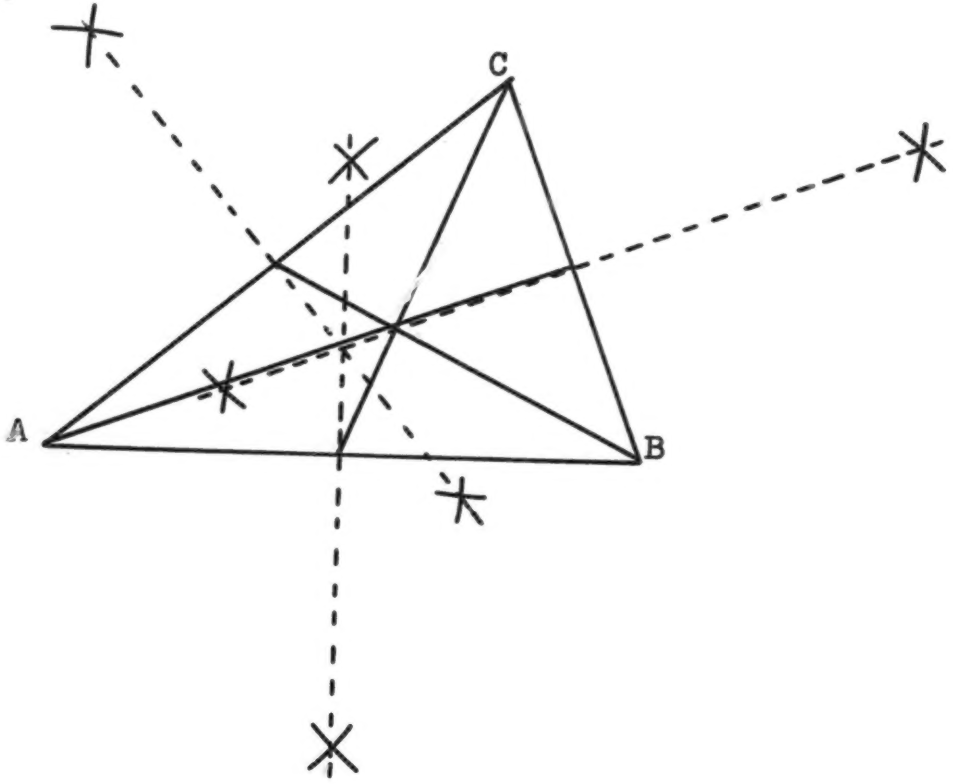
d. **Parallel**

9a.

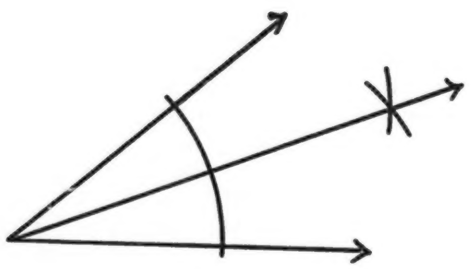


Review Exercises (continued)

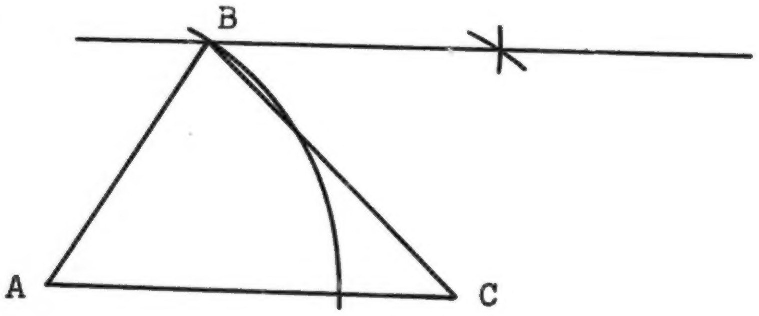
9b.



10.



11.



12. a, b
13. 15 miles

UNIT 5

Answer Key

Pages 1-4 Class Discussion-1

1. 1B
2. Flight 2 and Ship A
3. 2
4. 1C, 2C
5. 3 * 2A, 2B, 2C
6. 3 * 1A, 1B, 1C
7. 1A, 2A, 1B, 2B, 1C, 2C
8. 6
9. 3
10. 3A, 3B, 3C
11. 1A, 2A, 3A, 1B, 2B, 3B, 1C, 2C, 3C
12. 9
13. 1A, 2A, 3A, 4A,
    1B, 2B, 3B, 4B
14. 8 * See third row of table below.

Number of Flights	Number of Ships	Total Number of Routes
2	3	6
3	3	9
4	2	8
4	1	4

15. 4; see row 4 of table in exercise 14.
16. Yes. The number of flights that are available multiplied by the number of ships that are available is equal to the total number of routes.
17. 6
18. 6
19. 12
20. 18
21. Find $m \cdot n \cdot p$.

Pages 5-6 Exercises-1

1. 20
2. 280
3. S1
   S2
   S3
   T1
   T2
   T3
4. 6
5. 21
6. 30
7. RRR
   RRW
   RRG
   RWR
   RWW
   RWG
   RGR
   RGW
   RGG
   WWW
   WWR
   WWG
   WRR
   WRW
   WRG
   WGR
   WGW
   WGG
   GGG
   GGR
   GGW
   GRR
   GRW
   GRG
   GWR
   GWW
   GWG
8. HT
   HH
   TH
   TT
9. TTH
   TTT
   THT
   THH
   HHH
   HTH
   HHT
   HTT
10. 16
11.

Number of Tosses	Number of Possible Results
1	2
2	4
3	8
4	16
5	32
6	64

To find the number of different results, use 2 as a factor as many times as there are tosses and find the product. (The student may answer that

Exercises-1 (continued)

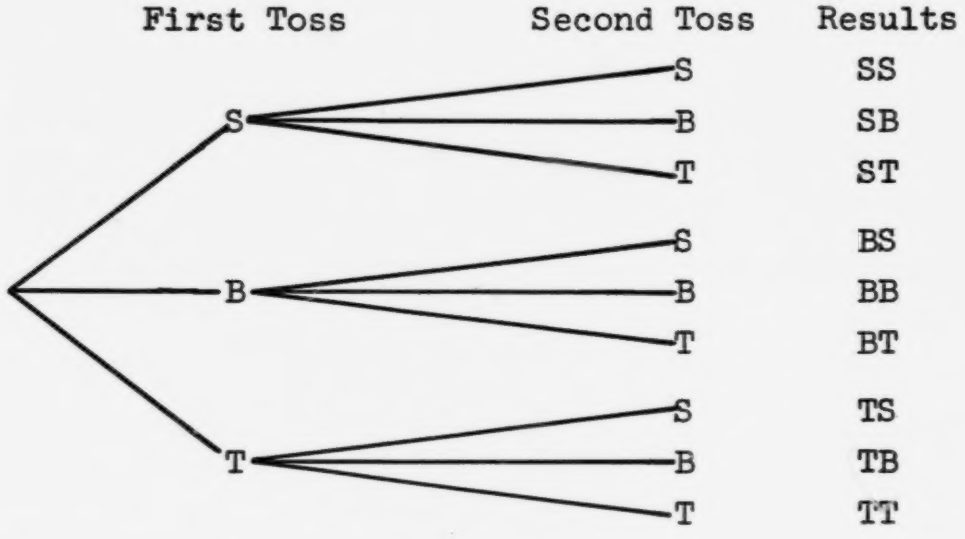
you double the number of different results obtained for the number of tosses which is one less than the number of tosses you are considering.)

Page 8 Class Discussion-2a

1. 3
2. 3
3.

Number of ways first cap can come to rest	Number of ways second cap can come to rest	Number of possible outcomes
3	x 3	= 9

4.


First Toss
Second Toss
Results
S
B
T
S
B
T
S
B
T
S
B
T
SS
SB
ST
BS
BB
BT
TS
TB
TT

Page 9 Class Discussion-2b

1. 12 *

Number of appetizers	Number of entrees	Number of desserts	Total number of completed dinners
2	x 3	x 2	= 12

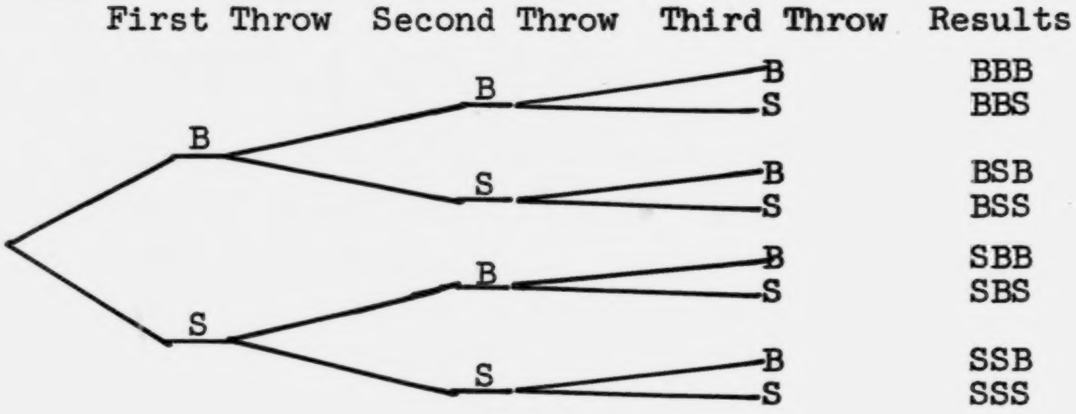
2. $m \cdot n \cdot p$
3. $m \cdot n \cdot p \cdot q$

Page 10 Exercises-2

1. 7 · 26 = 182.
2.

Number of Numerals Available for First Digit	Number of Numerals Available for Second Digit	Total Number of Possible Results
7 x	7 =	49

3. 64
4.



First Throw
Second Throw
Third Throw
Results
B
S
B
S
B
S
B
S
B
S
B
S
B
S
BBB
BBS
BSB
BSS
SBB
SBS
SSB
SSS

8 different results are possible.

5. 52,900
6. 676,000
7. Yes (1,757,600)
8. 1,081,600
9. 10,000,000
10. 9,000,000
11. Answer varies with state.

Pages 12-13 Class Discussion-3

1. 3
2. 2
3. 2
4. 2
5. 1
6. 1

Class Discussion-3 (continued)

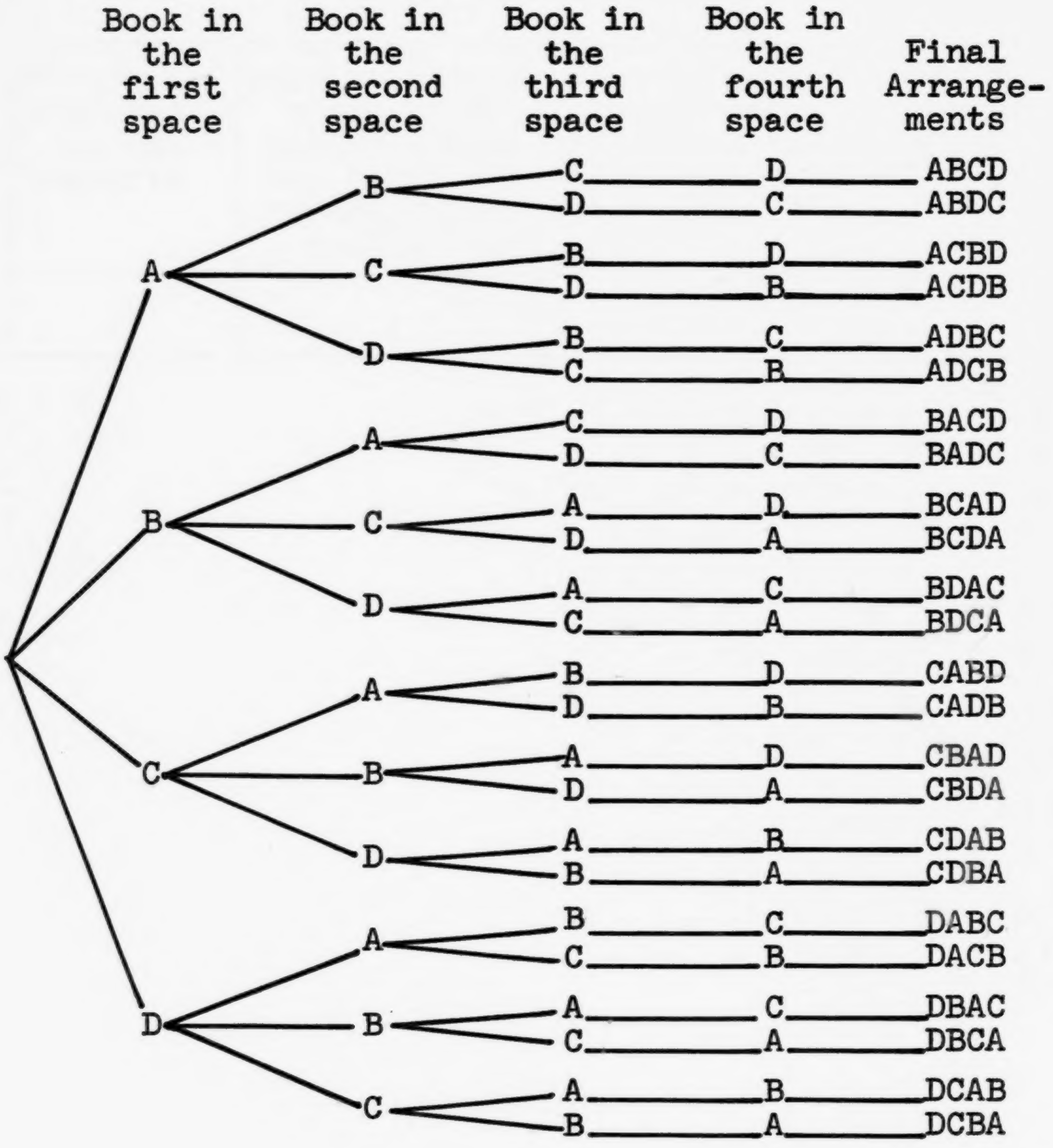
7.

Number of ways first space can be filled	Number of ways second space can be filled if first space is filled	Number of ways third space can be filled if first and second spaces are filled	Number of ways book can be arranged
3 x	2 x	1 =	6

8. 3
9. It is the same.
10. 2
11. It is the same.
12. 1 * Yes

Page 14 Exercises-3

1. 24 *


Book in the first space
Book in the second space
Book in the third space
Book in the fourth space
Final Arrangements
ABCD
ABDC
ACBD
ACDB
ADBC
ADCB
BACD
BADC
BCAD
BCDA
BDAC
BDCA
CABD
CADB
CBAD
CBDA
CDAB
CDBA
DABC
DACB
DBAC
DBCA
DCAB
DCBA

Exercises-3 (continued)

2.

Number of ways first space can be filled	Number of ways second space can be filled if first space is filled	Number of ways third space can be filled if first and second spaces are filled	Number of ways fourth space filled if first, second, third spaces are filled	Number of ways books can be arranged
4 x	3 x	2 x	1 =	24

3.

Number of ways first position may be filled	Number of ways second position may be filled after 1st is filled	Number of ways third position may be filled after 1st and 2nd are filled	Number of ways fourth position may be filled after 1st, 2nd and 3rd are filled	Number of ways fifth position may be filled after 1st, 2nd 3rd, and 4th are filled	Total number of ways cars may be arranged
5 x	4 x	3 x	2 x	1 =	120

4. Yes. (Each number in the sequence is one less than the previous number.)

5. $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

Exercises-3 (continued)

6.

Number of ways first position can be filled	Number of ways second position can be filled after 1st is filled	Number of ways third position can be filled after 1st, 2nd are filled	Number of ways fourth position can be filled after 1st, 2nd, 3rd are filled	Number of ways fifth position can be filled after 1st, 2nd, 3rd, 4th are filled	Number of ways sixth position can be filled after 1st, 2nd, 3rd, 4th, 5th are filled	Total number of ways cars can be arranged
6 x	5 x	4 x	3 x	2 x	1 =	720

7. 720
8. $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
9. $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3{,}628{,}800$.

Pages 15-16 Class Discussion-4

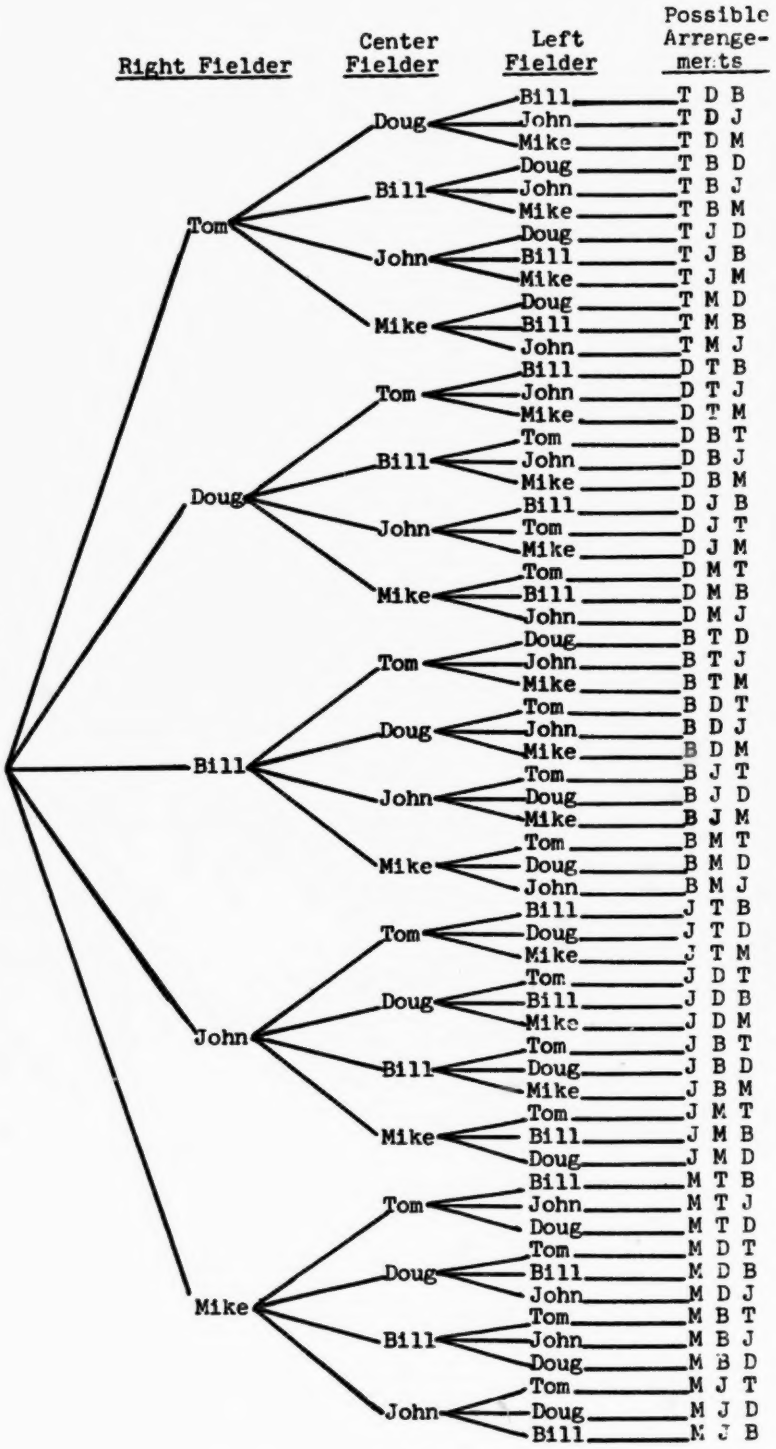
1. 12!
2. 85!
3. 37!
4. $2 \cdot 1$
5. $(2 \cdot 1) \cdot (3 \cdot 2 \cdot 1) = 12$.
6. 5
7. 20

Page 16 Exercises-4

1. 7!
2. 720
3. 5040
4. 40,320
5. 9! * 362,880

Pages 16-18 Class Discussion-5

1.


Right Fielder
Center Fielder
Left Fielder
Possible Arrangements
Tom
Doug
Bill T D B
John T D J
Mike T D M
Bill
Doug T B D
John T B J
Mike T B M
John
Doug T J D
Bill T J B
Mike T J M
Mike
Doug T M D
Bill T M B
John T M J
Doug
Tom
Bill D T B
John D T J
Mike D T M
Bill
Tom D B T
John D B J
Mike D B M
John
Bill D J B
Tom D J T
Mike D J M
Mike
Tom D M T
Bill D M B
John D M J
Bill
Tom
Doug B T D
John B T J
Mike B T M
Doug
Tom B D T
John B D J
Mike B D M
John
Tom B J T
Doug B J D
Mike B J M
Mike
Tom B M T
Doug B M D
John B M J
John
Tom
Bill J T B
Doug J T D
Mike J T M
Doug
Tom J D T
Bill J D B
Mike J D M
Bill
Tom J B T
Doug J B D
Mike J B M
Mike
Tom J M T
Bill J M B
Doug J M D
Mike
Tom
Bill M T B
John M T J
Doug M T D
Doug
Tom M D T
Bill M D B
John M D J
Bill
Tom M B T
John M B J
Doug M B D
John
Tom M J T
Doug M J D
Bill M J B

Class Discussion-5 (continued)

2. 5
3. 4
4. 3
5.

Number of ways to choose right fielder	Number of ways to choose center fielder	Number of ways to choose left fielder	Total number of ways outfield can be chosen
5 x	4 x	3 =	60

6. 60
7. 120
8. 840
9. 1680

Pages 17-18 Exercises-5

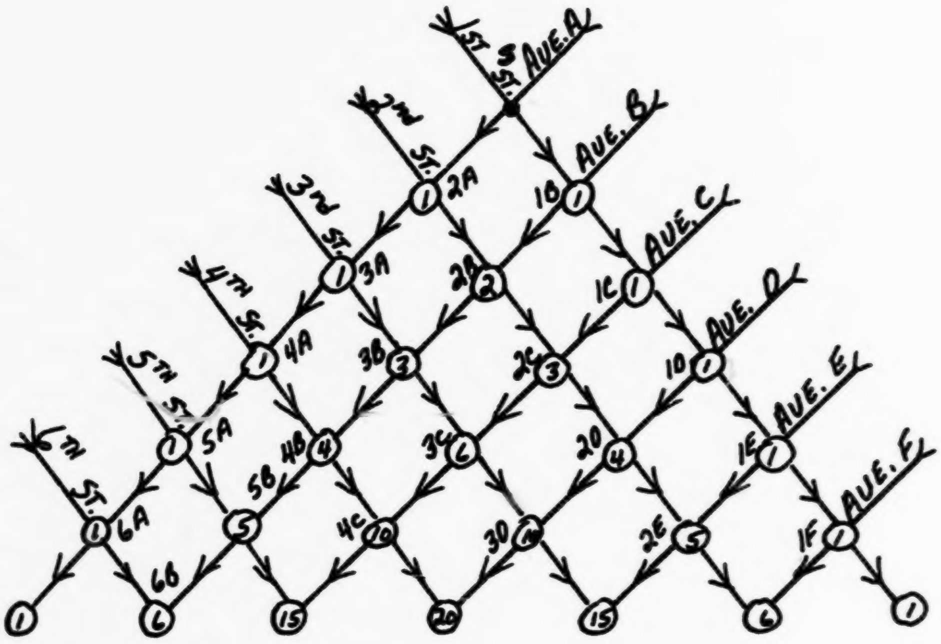
1. $12 \cdot 11 \cdot 10 = 1320.$
2. $13 \cdot 12 \cdot 11 \cdot 10 = 17{,}160.$
3. $4 \cdot 3 \cdot 2 \cdot 1 = 24.$
4. $5 \cdot 4 \cdot 3 \cdot 2 = 120.$
5. $4 \cdot 3 \cdot 2 = 24.$
6. $10 \cdot 9 \cdot 8 = 720.$
7. $5 \cdot 4 = 20.$
8. $7 \cdot 6 = 42.$
9. $30 \cdot 29 \cdot 28 = 24{,}360.$
10. $27 \cdot 26 \cdot 25 = 17{,}550.$
11. $4 \cdot 3 \cdot 2 \cdot 1 = 24.$
12. $3 \cdot 5 = 15.$
13. $6 \cdot 6 = 36.$

Page 20 Class Discussion-6

1. 1
2. 2
3. 1 * 1
4. 2B or 1C
5. 2
6. 1
7. No
8. Yes * 3
9. 3
10. 3B or 2C

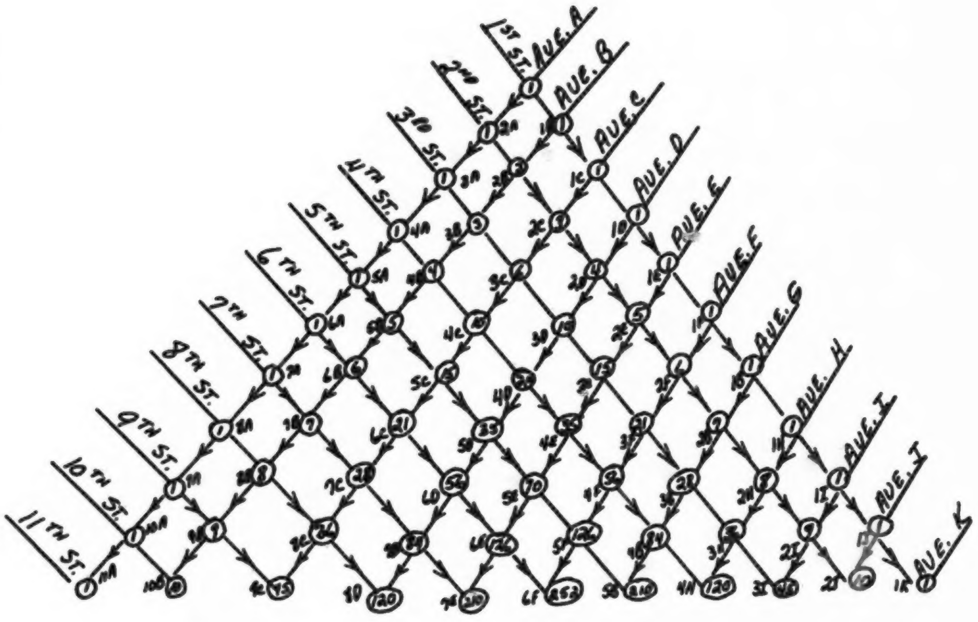
Class Discussion-6 (continued)

11. 6
12. 1 * 1 * 1
13. 2
14. 1
15.


1ST ST.
AVE. A
2ND ST.
AVE. B
3RD ST.
AVE. C
4TH ST.
AVE. D
5TH ST.
AVE. E
6TH ST.
AVE. F
2A
1B
3A
2B
1C
4A
3B
2C
1D
5A
4B
3C
2D
1E
6A
5B
4C
3D
2E
1F
6B
1
1
1
2
1
1
3
3
1
1
4
6
4
1
1
5
10
10
5
1
1
6
15
20
15
6
1

Page 22 Exercises-6

1.


1ST ST.
2ND ST.
3RD ST.
4TH ST.
5TH ST.
6TH ST.
7TH ST.
8TH ST.
9TH ST.
10TH ST.
11TH ST.
AVE. A
AVE. B
AVE. C
AVE. D
AVE. E
AVE. F
AVE. G
AVE. H
AVE. I
AVE. J
AVE. K
1
10
45
120
210
252
210
120
45
10
1

2a. 5 6 7 8
b. 15 21 28 36
c. 35 56 84 120
d. 70 126 210 330

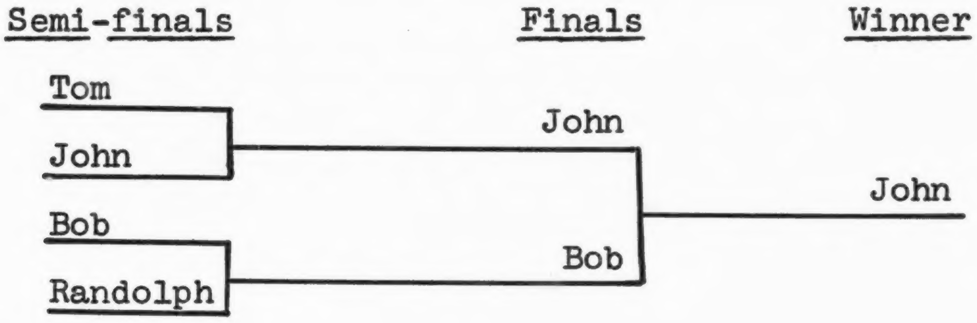
3. The same sequence * Add one to a member of the sequence to obtain its successor.
4. The same sequence * Add the consecutive integers 2, 3, 4, ... to the 1st, 2nd, 3rd, ... members of the sequences to obtain their successors.
5. The same sequence
6. 5th Street
7. Look down 6th St. or Avenue F and extend the map. The next four numbers are 126, 252, 462, 792.
8. 252
9. 462

Page 24 Class Discussion-7

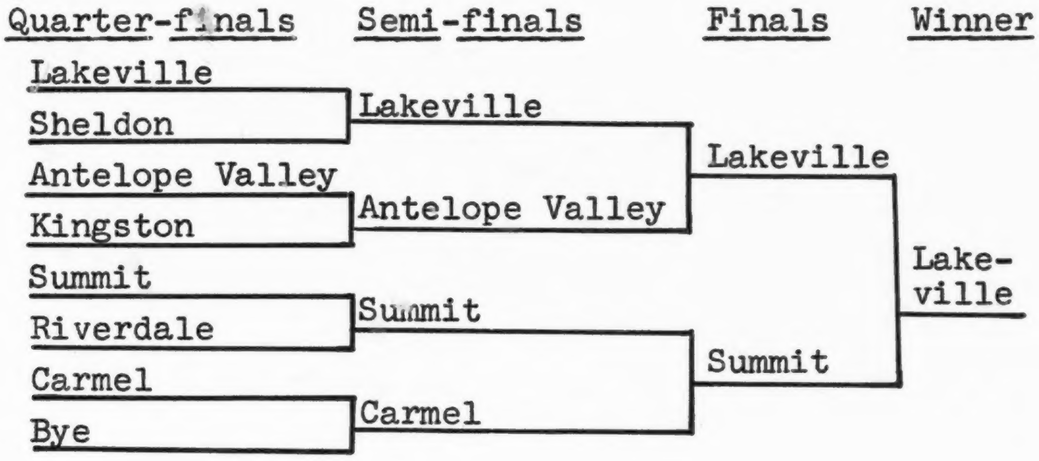
1. 1, 2, 4, 8, 16, 32, 64
2. 1, 2, 4, 8, 16, 32, 64
3. Yes * 128, 256, 512, ...

Pages 24-27 Exercises-7

1. 5 * 5 * The same
2. 10 * 10
3. 20 * The same
4. Answers vary.
5. 1, 3, 6, 10, 15 * Yes. On 3rd St. or Ave. C * 21
6. 1, 4, 10, 20, 35 * Yes * On 4th St. or Ave. D
7. 5th St. or Ave. E
8.


Semi-finals
Finals
Winner
Tom
John
Bob
Randolph
John
Bob
John

9. 2 * 4 * 8
10. 2 * 3
11. 32
12. 3 byes * 4 rounds (To find the number of byes subtract the number of players from the next greater counting number that is a power of two; 16-13=3.)
13. 10
14. 6
15. 6
16.


Quarter-finals
Semi-finals
Finals
Winner
Lakeville
Sheldon
Antelope Valley
Kingston
Summit
Riverdale
Carmel
Bye
Lakeville
Antelope Valley
Summit
Carmel
Lakeville
Summit
Lake-
ville

Pages 28-30 Class Discussion-8a

1. Ann and Cathy, Betty and Cathy
2. 3
3. No
4. 6
5. 2 times as many * There are two arrangements for each selection.
6. 12 * 6
7. Yes * Different representations of the same selection
8. 2 times as many
9.

Number of Ways of Filling Each Seat				
Left		Right		Number of Arrangements
4	x	3	=	12

Pages 30-32 Class Discussion-8b

1. $4 \cdot 3 \cdot 2 = 24$
2. Yes
3. 4
4. {Ann, Betty, Cathy}, {Ann, Cathy, Diane}, {Ann, Betty, Diane}, {Betty, Cathy, Diane}
5. 6
6. 6
7. 6
8. 6
9. Divide the total number of possible arrangements by the number of possible arrangements in each selection.

Page 33 Exercises-8

1. 3
2. $(3 \cdot 2) \div (2 \cdot 1) = 3.$
3. $(4 \cdot 3) \div (2 \cdot 1) = 6.$
4. $(4 \cdot 3 \cdot 2) \div (3 \cdot 2 \cdot 1) = 4.$
5. $(5 \cdot 4 \cdot 3) \div (3 \cdot 2 \cdot 1) = 10.$
6. $(6 \cdot 5 \cdot 4 \cdot 3) \div (4 \cdot 3 \cdot 2 \cdot 1) = 15.$
7. $(6 \cdot 5 \cdot 4) \div (3 \cdot 2 \cdot 1) = 20.$
8. $(5 \cdot 4) \div (2 \cdot 1) = 10.$
9. 1
10. $(7 \cdot 6 \cdot 5) \div (3 \cdot 2 \cdot 1) = 35.$

Exercises-8 (continued)

11. $(5 \cdot 4 \cdot 3 \cdot 2) \div (4 \cdot 3 \cdot 2 \cdot 1) = 5.$
12. $(8 \cdot 7 \cdot 6 \cdot 5 \cdot 4) \div (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) = 56.$
13. $(6 \cdot 5) \div (2 \cdot 1) = 15.$
14. $(8 \cdot 7 \cdot 6) \div (3 \cdot 2 \cdot 1) = 56.$
15. $(7 \cdot 6) \div (2 \cdot 1) = 21.$

Pages 35-37 Class Discussion-9

1.

0 heads	1 head	2 heads
1	2	1

2.

0 heads	1 head	2 heads	3 heads
1	3	3	1

3.

0 heads	1 head	2 heads	3 heads	4 heads
1	4	6	4	1

4. 1, 4, 6, 4, 1
5. 1, 5, 10, 10, 5, 1
6. Yes. The rows of Pascal's triangle
7. 3 * 3 * 1

Committee of		
One	Two	Three
3	3	1

(In the questions here and on page 33, we are selecting, first, 1 object from among 3 objects and, second, 2 objects from among 3 objects.)

Pages 37-39 Exercises-9

1a. 4
b. 6
c. 4
d. 1
2a. 5
b. 10

Exercises-9 (continued)

2c. 10
d. 5
e. 1
3. Yes (the rows of Pascal's triangle with the first number of each omitted)

Number of Members	1	2	3	4	5	6
Number of ways the Committee can be formed	6	15	20	15	6	1

4. (See 1st 5 rows of answer to exercise 5.)
5.

Table for 1 coin	1 1
Table for 2 coins	1 2 1
Table for 3 coins	1 3 3 1
Table for 4 coins	1 4 6 4 1
Table for 5 coins	1 5 10 10 5 1
Table for 6 coins	1 6 15 20 15 6 1
Table for 7 coins	1 7 21 35 35 21 7 1
Table for 8 coins	1 8 28 56 70 56 28 8 1
Table for 9 coins	1 9 36 84 126 126 84 36 9 1
Table for 10 coins	1 10 45 120 210 252 210 120 45 10 1

6a. 5th row
b. 6th row
c. 7th row
d. 10th row

Exercises-9 (continued)

7. Yes * The number of members in the club corresponds to the row number. The numbers in that row correspond in order to committees of various sizes, omitting the first number in the row.

8. 6 member club:

	Committee of					
No Members	One	Two	Three	Four	Five	Six
1	6	15	20	15	6	1

9. 7 member club:

	Committee of						
No Members	One	Two	Three	Four	Five	Six	Seven
1	7	21	35	35	21	7	1

8 member club:

	Committee of							
No Members	One	Two	Three	Four	Five	Six	Seven	Eight
1	8	28	56	70	56	28	8	1

9 member club:

	Committee of								
No Members	One	Two	Three	Four	Five	Six	Seven	Eight	Nine
1	9	36	84	126	126	84	36	9	1

Exercises-9 (continued)

9. 10 member club:

	Committee of									
No Members	One	Two	Three	Four	Five	Six	Seven	Eight	Nine	Ten
1	10	45	122	210	252	210	122	45	10	1

10a. 56
b. 210
c. 84
d. 15
e. 84

Pages 40-41 Class Discussion-10a

(N.B. The probability of a successful outcome

$$= \frac{\text{Number of possible successful outcomes}}{\text{Number of possible outcomes}})$$

1. $\frac{2}{5}$
2. $\frac{3}{10}$
3. 10
4. 3
5. $\frac{5}{12}$
6. $\frac{1}{2}$
7. $\frac{6}{15}$ or $\frac{2}{5}$
8. $\frac{9}{15}$ or $\frac{3}{5}$

Pages 41-43 Class Discussion-10b

1. HH, HT, TH, TT
2. One * $\frac{1}{4}$
3. $\frac{1}{2}$
4. $\frac{1}{4}$
5. 8
6. 1
7. $\frac{1}{8}$

Class Discussion-10b (continued)

8. $\frac{3}{8}$

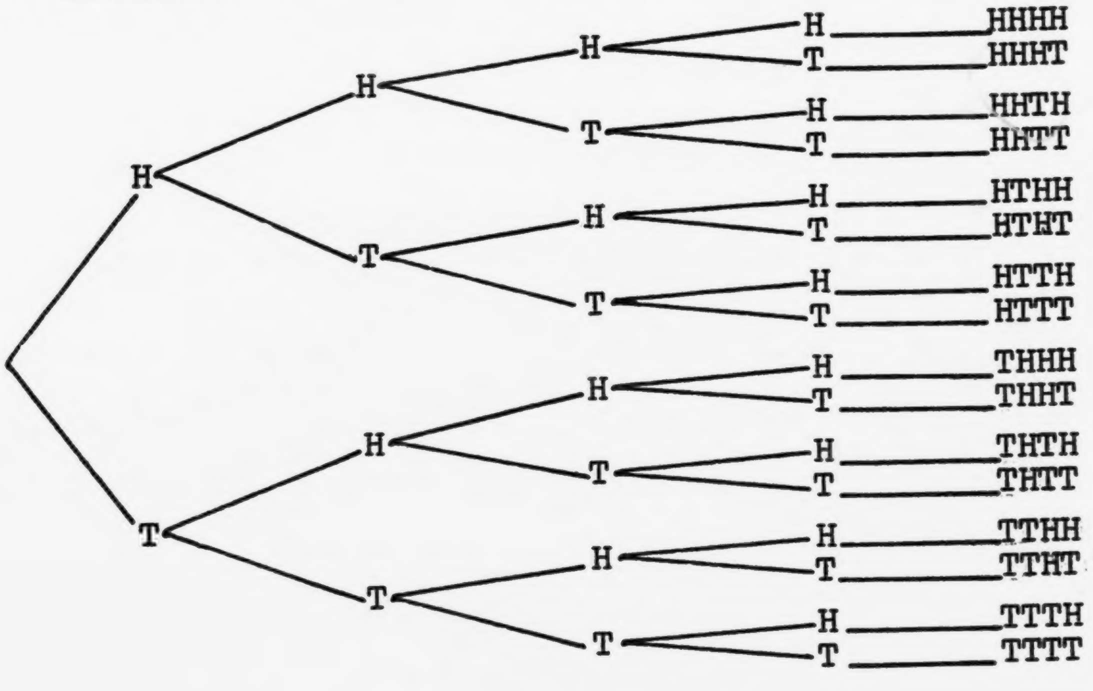
9. $\frac{3}{8}$

10.

Number of heads	Probability
0	$\frac{1}{8}$
1	$\frac{3}{8}$
2	$\frac{3}{8}$
3	$\frac{1}{8}$

Pages 43-44 Exercises-10

1. 1st toss 2nd toss 3rd toss 4th toss Results



H
T
H
T
H
T
H
T
H
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H
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H
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T
H
T
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HHHH
HHHT
HHTH
HHTT
HTHH
HTHT
HTTH
HTTT
THHH
THHT
THTH
THTT
TTHH
TTHT
TTTH
TTTT

Exercises-10 (continued)

2. 16
3. 4
4. $\frac{4}{16}$ or $\frac{1}{4}$
5. $\frac{6}{16}$ or $\frac{3}{8}$

6a. 64
b. $\frac{15}{64}$
c. $\frac{6}{64}$ or $\frac{3}{32}$

7. $\frac{21}{128}$
8. $\frac{8}{13}$
9. $\frac{4}{20}$ or $\frac{1}{5}$ * $\frac{6}{20}$ or $\frac{3}{10}$ * $\frac{10}{20}$ or $\frac{1}{2}$
10. $\frac{5}{26}$ * $\frac{9}{26}$

Pages 46-47 Class Discussion-11

1. B seems more likely because the cap is wider at the bottom. (Answers vary.)
2. S seems more likely. (Answers may vary.)
3. S, B, T (Answers may vary.)
4. Answers vary.
5. a-c, e-g. Answers depend on results of individual experiments.
   d. They are called experimental probabilities because they are obtained by examining the results of an experiment.

Page 49 Review Exercises

1. HHHH
HHHT
HHTH
HHTT
HTHH
HTHT
HTTH
HTTT
TTTT
TTTH
TTHT
TTHH
THTT
THTH
THHT
THHH

2. $5 \cdot 5 = 25$.
3. $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40{,}320$
4.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Review Exercises (continued)

4a. 20

b. $3 * 6$

c. 64

d. $\frac{20}{64}$ or $\frac{5}{16}$

5. $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.

6. $(12 \cdot 11) \div (2 \cdot 1) = 66$.

7. $(6 \cdot 5 \cdot 4) \div (3 \cdot 2 \cdot 1) = 20$.

8. $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

9a. 4

b. 15

c. 5

d. 6

10. $(6 \cdot 5) \div (2 \cdot 1) = 15$.

11. $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

12. $\frac{10}{22}$ or $\frac{5}{11}$

13. $\frac{6}{16}$ or $\frac{3}{8}$

14. $\frac{15}{1000}$ or $\frac{3}{200}$